A New Approach to Construct Virtual Array With Increased Degrees of Freedom for Moving Sparse Arrays

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***Abstract*—In this letter, a novel approach to construct virtual array is proposed by exploiting synthetic aperture technology and the concept of difference coarray for sparse arrays. First, we prove that the proposed method can increase the degrees of freedom (DOF) of an arbitrary moving sparse array threefold. Therefore, the maximal number of resolvable sources for direction-of-arrival (DOA) estimation is tripled as well. Furthermore, we prove that the difference coarray of synthetic arrays is a hole-free uniform linear array (ULA) if the difference coarray of original arrays is hole-free. Therefore, for ULA based DOA estimation methods, better performance can be achieved by fully employing all elements in the difference coarray of synthetic arrays. In addition, unlike the dilated nested array method, which enlarges the intersensor spacing of nested arrays threefold to triple DOF, the proposed method does not need to increase array aperture physically. Thus it is more practical to mount a sensor array on a moving platform when taking space limitation into consideration. Simulation results are given to demonstrate the superiority of the proposed method.**

***Index Terms*—Sparse arrays, degrees of freedom (DOF), virtual arrays, moving arrays, direction-of-arrival (DOA) estimation.**

I. Introduction

ARRAY signal processing plays a vital role in various applications such as radar [1], sonar [2] and source localization [3]–[5]. The utilization of sensor arrays enables us easily to extract source profiles such as direction-of-arrival (DOA). In array processing society, virtual array design is always a hot topic due to the ability to detect more sources than sensors. In the past decades, several typical methods have been proposed to construct virtual arrays including augmented matrix methods [6]–[8], cumulant-based methods [9]–[11], Khatri-Rao (KR) methods [12] and synthetic aperture methods [13]–[15].

By employing redundancy of covariance matrix of uniform linear arrays (ULAs), the augmented matrix based methods construct an augmented covariance matrix to perform DOA estimation [6]–[8]. However, the methods are based on minimum redundancy arrays (MRAs) [16], which have no closed-form expression for array geometry. Generally, computer search is needed to find sensor positions. Thus, these methods are impractical especially for a large number of sensors.

Through the use of high order statistics, the cumulant-based methods can construct a large virtual array and significantly increase degrees of freedom (DOF) [9]–[11]. However, the cumulant-based methods are inapplicable for Gaussian sources. Besides, these approaches suffer from high computational cost.

The KR-based method is one of the most commonly used ways to obtain virtual arrays nowadays [12], [17]. In KR space, a difference coarray is obtained so that DOF can be increased substantially. Inspired by the concept of difference coarray, several types of sparse arrays have been proposed to further increase DOF or mitigate mutual coupling including co-prime arrays (CPAs) [18]–[20], nested arrays (NAs) [17], super nested arrays [21], [22], augmented nested arrays (ANAs) [23], improved nested arrays [24], generalized nested arrays [25], MISC arrays [26] and Cantor arrays [27]. These sparse arrays achieve much more DOF than ULAs.

In many cases, sensor arrays are placed on a moving platform. For example, a group of hydrophones, also known as towed arrays, are towed behind a ship or submarine to detect other ships or marine animals [28]. For moving arrays, virtual arrays can be constructed by utilizing synthetic aperture technology [13]–[15].

Recently, synthetic aperture technology and difference coarray concept are combined to yield a larger virtual ULA when sparse arrays move [29]–[33]. In [29]–[31], the missing elements of the difference coarray of a CPA are filled in order to obtain a fully filled ULA through several synthetic steps. In [32], [33], consecutive spatial correlation lags for certain types of sparse arrays are derived through one synthetic step. Since some missing holes in the difference coarray are filled, a larger ULA is obtained in [29]–[33]. Consequently, the number of sources that can be identified is increased. However, there are two issues. First, the DOF increase is quite limited. Especially for the sparse arrays that already have hole-free difference coarrays such as NAs, DOF is only increased by 2. Second, all existing methods on a moving platform can only increase DOF for certain types of sparse arrays but not for all. In [34], to deal with the first problem, the dilated NAs have been proposed by enlarging the intersensor spacing of NAs threefold. Accordingly, DOF is tripled as well.

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However, the second issue still exists. Furthermore, on a moving platform, it is usually impractical to increase the intersensor spacing threefold due to space limitation.

In this letter, a novel approach to construct virtual array for a moving array is proposed to tackle the two issues above. First, we prove that the proposed method can triple DOF for an arbitrary sparse array through a single synthetic step. Thus, the maximal number of detectable sources is increased threefold as well. Moreover, we prove that the difference coarray of synthetic arrays is a hole-free ULA if the difference coarray of original arrays is hole-free. Therefore, for ULA based DOA estimation methods, better performance can be achieved by fully employing all elements in the difference coarray of synthetic arrays. Besides, unlike the dilated NAs method, which triples the intersensor spacing of NAs, the proposed method does not need to increase the interelement spacing of sparse arrays. Therefore, it is more feasible to mount a sparse array on any moving platform due to space limitation. Simulation results show that the proposed method outperforms the state-of-the-art method in [33].

In the letter, the superscript "T" denotes the transpose of a vector or a matrix. The minimum and maximum elements of a set $\mathbb{N}$ are denoted by $\min(\mathbb{N})$ and $\max(\mathbb{N})$, respectively.

II. Problem Formulation

Consider the scenario that K far-field narrowband sources impinge on a sparse array with M sensors and underlying intersenosr spacing d. The position of the i-th sensor is denoted by $n_i d$, where n_i is an integer. The array is assumed to move at a slow speed v along the direction of layout of the sparse array so that the DOAs of the sources $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_K]^T$ can be considered unchanged during a short processing time. Let $\boldsymbol{x}(t) = [x_1(t), \ldots, x_M(t)]^T$ denote the array output at time t, where $x_i(t)$ is the received signal of the i-th sensor, $i = 1, \ldots, M$. Then, $\boldsymbol{x}(t)$ can be expressed as [33]

$$\boldsymbol{x}(t) = \mathbf{A}\boldsymbol{s}(t) + \boldsymbol{w}(t), \tag{1}$$

where $\mathbf{A} = [\boldsymbol{a}(\bar{\theta}_1), \ldots, \boldsymbol{a}(\bar{\theta}_K)]$ denotes the array manifold matrix, $\boldsymbol{a}(\bar{\theta}_k) = [\exp(-j2\pi n_1\bar{\theta}_k), \ldots, \exp(-j2\pi n_M\bar{\theta}_k)]^T$ is the steering vector, $\bar{\theta}_k = d\sin(\theta_k)/\lambda$ refers to the normalized DOA of the k-th source, λ represents the wavelength of the narrowband sources, $\boldsymbol{s}(t) = [s_1(t)\exp(-j2\pi vt\bar{\theta}_1/d), \ldots, s_K(t)\exp(-j2\pi vt\bar{\theta}_K/d]^T$ is the source vector and $\boldsymbol{w}(t) = [w_1(t), \ldots, w_M(t)]^T$ stands for the noise vector received by sensors, respectively.

At time $t+\tau$, where τ denotes a time interval, the received signal vector becomes

$$\boldsymbol{x}(t+\tau) = \mathbf{A}\boldsymbol{s}(t+\tau) + \boldsymbol{w}(t+\tau). \tag{2}$$

Let $\bar{\mathbf{A}} = [\bar{\boldsymbol{a}}(\bar{\theta}_1), \bar{\boldsymbol{a}}(\bar{\theta}_2), \ldots, \bar{\boldsymbol{a}}(\bar{\theta}_K)]$ with

$$\bar{\boldsymbol{a}}(\bar{\theta}_k) = \boldsymbol{a}(\bar{\theta}_k)\exp(-j2\pi v\tau\bar{\theta}_k/d). \tag{3}$$

Under the assumption of narrowband sources, we can obtain

$$\bar{\boldsymbol{x}}(t) = \bar{\mathbf{A}}\boldsymbol{s}(t) + \bar{\boldsymbol{w}}(t), \tag{4}$$

where $\bar{\boldsymbol{x}}(t) = \boldsymbol{x}(t+\tau)\exp(-j2\pi f\tau)$ and $\bar{\boldsymbol{w}}(t) = \boldsymbol{w}(t+\tau)\exp(-j2\pi f\tau)$. Compared (4) with (1), $\bar{\boldsymbol{x}}(t)$ behaves like the received signal vector of a new array, which is the shifted version of the physical array with a displacement of $v\tau$.

Combining (1) and (4), yields the synthetic received signal

$$\boldsymbol{y}(t) = \begin{bmatrix} \boldsymbol{x}(t) \\ \bar{\boldsymbol{x}}(t) \end{bmatrix} = \mathbf{B}\boldsymbol{s}(t) + \boldsymbol{v}(t), \tag{5}$$

where $\mathbf{B} = [\boldsymbol{b}(\bar{\theta}_1), \boldsymbol{b}(\bar{\theta}_2), \ldots, \boldsymbol{b}(\bar{\theta}_K)]$ with $\boldsymbol{b}(\bar{\theta}_k) = [\boldsymbol{a}^T(\bar{\theta}_k), \bar{\boldsymbol{a}}^T(\bar{\theta}_k)]^T$ denotes the array manifold matrix of the synthetic array and $\boldsymbol{v}(t) = [\boldsymbol{w}^T(t), \bar{\boldsymbol{w}}^T(t)]^T$ represents the measured noise of the synthetic array, respectively. According to the steering vector $\boldsymbol{b}(\bar{\theta})$, the synthetic array is composed of the original physical array and its shifted version. Note that the difference coarray of the synthetic array can be obtained to further increase DOF in the KR space [12], [17].

Here, the problem is to choose τ to obtain a hole-free difference coarray with high DOF for DOA estimation, or at least, to gain a large contiguous part around origin, i.e., a large central ULA in the difference coarray of the synthetic array.

III. The Proposed Construction Method

For simplicity, we use the set of n_i, which represents the position of i-th sensor, to denote the sensor array after normalized by underlying intersensor spacing d. Therefore, the physical array can be denoted by $\mathbb{N} = \{n_1, n_2, \ldots, n_M\}$.

For a sparse array $\mathbb{N} = \{n_i | n_i, 1 \le i \le M\}$, its difference coarray is defined as [17], [21], [35]

$$\mathbb{D} = \{n_i - n_j | i, j = 1, 2, \ldots, M\}. \tag{6}$$

We use the set $\mathbb{U}$ to denote the largest contiguous part around zero in the difference coarray $\mathbb{D}$[21], i.e.,

$$\mathbb{U}: \text{the largest ULA around zero in } \mathbb{D}. \tag{7}$$

$\mathbb{U} = \mathbb{D}$ if the sparse array has a hole-free difference coarray. The numbers of sensors in $\mathbb{D}$ and $\mathbb{U}$ are termed DOF and uniform DOF (uDOF), respectively [21].

In order to significantly increase uDOF, in this letter, we propose to choose the array synthesis time τ that satisfies

$$\tau = Dd/v = (2P+1)d/v, \tag{8}$$

where P is given by the maximum integer in $\mathbb{U}$, i.e.,

$$P = \max\mathbb{U}. \tag{9}$$

Note that $D = 2P+1$ is exactly the number of elements in $\mathbb{U}$, which is uDOF of the sparse array. The uDOF decides the number of sources that can be identified when using typical ULA based methods [36], [37] to perform DOA estimation.

Plugging (8) into (3), we obtain the shifted array of $\mathbb{N}$ with a displacement $v\tau = (2P+1)d$ as

$$\bar{\mathbb{N}} = \{\bar{n}_i | \bar{n}_i = n_i + 2P + 1, 1 \le i \le M\}. \tag{10}$$

Since the synthetic array is a union set of $\mathbb{N}$ and $\bar{\mathbb{N}}$, it can be expressed as

$$\mathbb{S} = \mathbb{N} \cup \bar{\mathbb{N}}. \tag{11}$$

Then, we have the following theorem regarding the uDOF.

Theorem 1: Given a sparse array $\mathbb{N} = \{n_i | n_i, 1 \le i \le M\}$ and its shifted array $\bar{\mathbb{N}} = \{\bar{n}_i | \bar{n}_i = n_i + 2P + 1, 1 \le i \le M\}$

generated by τ in (8), the difference coarray of $\mathbb{S} = \mathbb{N} \cup \bar{\mathbb{N}}$ has a large contiguous part around zero, i.e., a ULA with $6P+3$ elements ranging from $-(3P+1)$ to $3P+1$.

Proof: First, according to the known conditions on P, the physical array $\mathbb{N}$ has a ULA with elements from $-P$ to P in its difference coarray. It indicates for $1 \le i, j \le M$ and $p = -P, -P+1, \ldots, P$, we can find a pair (i, j) such that

$$n_i - n_j = p, \quad p = -P, -P+1, \ldots, P. \tag{12}$$

In the synthetic array $\mathbb{S}$, for any $n_i, i = 1, 2, \ldots, M$, we can find a corresponding $\bar{n}_i$ such that

$$n_i = \bar{n}_i - (2P+1), i = 1, 2, \ldots, M. \tag{13}$$

Substituting (13) into (12), leads to

$$\bar{n}_i - n_j = p + 2P + 1, p = -P, -P+1, \ldots, P. \tag{14}$$

Note that $p + 2P + 1$ ranges from $P+1$ to $3P+1$ when p varies from $-P$ to P. Thus, (14) indicates that the difference coarray of the synthetic array $\mathbb{S}$ has a ULA with elements between $P+1$ and $3P+1$.

Moreover, according to (12) and (14), we deduce that the difference coarray of $\mathbb{S}$ has no hole at the range from 0 to $3P+1$. Due to the symmetric property, we conclude that the difference coarray of $\mathbb{S}$ has a ULA with $6P+3$ elements from $-(3P+1)$ to $3P+1$. Thus, Theorem 1 has been proven. ∎

Remark 1: From Theorem 1, the number of sensors in the largest ULA of the difference array is $6P+3 = 3D$. Thus, by leveraging array motion, uDOF is increased threefold.

Remark 2: Theorem 1 is applicable for arbitrary sparse arrays including CPAs, NAs, ANAs, MRAs and so forth.

Remark 3: Since $\max(\mathbb{S}) - \min(\mathbb{S}) = \max(\mathbb{N}) - \min(\mathbb{N}) + 2P + 1$, the aperture of the virtual coarray is $2(\max(\mathbb{S}) - \min(\mathbb{S})) = A_f + 4P + 2$, where $A_f = 2(\max(\mathbb{N}) - \min(\mathbb{N}))$ denotes the aperture of the difference coarray of $\mathbb{N}$. The effective aperture of difference coarray, which is the aperture of the largest ULA in the difference coarray, has increased from $2P$ to $6P+2$.

Remark 4: The DOF of the synthetic array $\mathbb{S}$ depends on the corresponding sparse array $\mathbb{N}$. Define two integer subsets of $\mathbb{D}$, i.e., $\mathbb{Z}_1$ and $\mathbb{Z}_2$, which include the elements of $\mathbb{D}$ in the range $[P+2, 3P+1]$ and $[3P+2, +\inf)$, respectively. The numbers of sensors in $\mathbb{Z}_1$ and $\mathbb{Z}_2$ are Q_1 and Q_2, respectively. According to the proof of Theorem 1, the elements in $\mathbb{Z}_1$ and $\mathbb{Z}_2$ produce the corresponding Q_1 and Q_2 elements in the difference coarray of $\mathbb{S}$. Note that the corresponding Q_1 elements in the difference coarray of $\mathbb{S}$ may overlap with the elements in $\mathbb{Z}_2$. Let Q_3 be the number of overlapped sensors in $\mathbb{Z}_2$. Then, the DOF of $\mathbb{S}$ is $6P + 3 + 2(Q_1 + 2Q_2 - Q_3)$.

In summary, for a sparse array $\mathbb{N}$, the method of the virtual array construction is given as follows.

(S1) Use (6) to compute $\mathbb{D}$;
(S2) Find the largest central ULA in $\mathbb{D}$, i.e., $\mathbb{U}$;
(S3) Obtain P by using (9);
(S4) Obtain the shifted array $\bar{\mathbb{N}}$ by exploiting (10);
(S5) Use (11) to obtain the synthetic array $\mathbb{S}$;
(S6) At last, the virtual array is obtained by computing the difference coarray of the synthetic array $\mathbb{S}$.

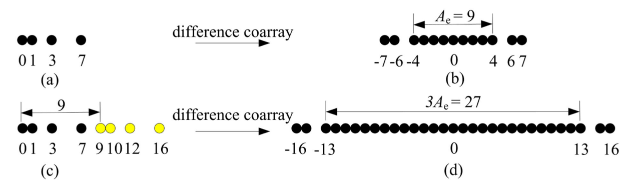

Fig. 1. An example of Theorem 1. (a) a sparse array, (b) the difference coarray of the sparse array, (c) the synthetic array, (d) the difference coarray of the synthetic array. In the synthetic array, black circles denote physical sensors while yellow circles denote synthesizing sensors. The uDOF that the proposed method achieves is tripled compared to that of the sparse array.

To show the method of the virtual array construction, we give an example in Fig. 1. The physical array $\{0, 1, 3, 7\}$ and its difference coarray $\mathbb{D} = \{0, \pm1, \pm2, \pm3, \pm4, \pm6, \pm7\}$ are shown in Fig. 1(a) and Fig. 1(b), respectively. Then we have $\mathbb{U} = \{0, \pm1, \pm2, \pm3, \pm4\}$. So $P = \max(\mathbb{U}) = 4$ and $D = 9$. Next, we obtain its synthetic array $\{0, 1, 3, 7, 9, 10, 12, 16\}$, shown in Fig. 1(c). Eventually, the difference coarray of the synthetic array is acquired, as shown in Fig. 1(d). Note that the difference coarray of the synthetic array has a ULA with 27 sensors from -13 to 13. Thus, uDOF is tripled.

Corollary 1: Given a sparse array $\mathbb{N} = \{n_i | n_i, 1 \le i \le M\}$ and its shifted array $\bar{\mathbb{N}} = \{\bar{n}_i | \bar{n}_i = n_i + 2P + 1, 1 \le i \le M\}$ generated by τ in (8), the difference coarray of the synthetic array $\mathbb{S} = \mathbb{N} \cup \bar{\mathbb{N}}$ is a hole-free ULA if $\mathbb{N}$ has a hole-free difference coarray.

Proof: Assume $\min(\mathbb{N}) = 0$. Since the difference coarray of $\mathbb{N}$ is hole-free, we obtain $\mathbb{U} = \mathbb{D}$. It indicates $\max(\mathbb{D}) = \max(\mathbb{N}) = P$. According to Theorem 1, the difference coarray of $\mathbb{S}$ has a ULA with $6P+3$ sensors ranging from $-(3P+1)$ to $3P+1$. Meanwhile, for $\mathbb{S}$, we have $\min(\mathbb{S}) = 0$ and $\max(\mathbb{S}) = \max(\mathbb{N}) + 2P + 1 = 3P + 1$. It implies that there is no element beyond the range $[-(3P+1), 3P+1]$ in the difference coarray of $\mathbb{S}$. Thus, Corollary 1 has been proven. ∎

Corollary 1 is useful in that if we expect to obtain a uniform linear coarray what we need to do is only to choose a sparse array whose difference coarray is hole-free.

Note that the distance that sparse arrays need to traverse in our method is $(2P+1)d$. Thus, the integer P and the number of sensors cannot be too large such that the DOA can be assumed unchanged. The specific value P depends on applications. Here, we give an example to show how large the integer P can be. Consider that a far-field acoustic source with $f = 1.5$ kHz impinges on a towed array in an ocean. Assume that the source-receiver range $r = 2$ km and the allowable DOA change amount $\Delta = 1°$. The propagation speed of acoustic wave is 1500 m/s. Then the maximal moving distance can be approximated as $\pi r \Delta/180 \approx 34.9$ m. We have $d = \lambda/2 = 0.5$ m. Thus, we obtain the maximum $P = 34$. It is large enough for many applications since the sparse arrays used in those applications do not have a large number of sensors in practice considering computational complexity and hardware cost.

To form synthetic array snapshots, the synthesis time τ should not exceed the signal coherence time, which is defined as the time by which the signal becomes uncorrelated. It has been shown that the theoretical coherence time can be up to tens of minutes [31],

TABLE I
THE uDOF FOR DIFFERENT CASES AND ARRAYS. THE PROPOSED METHOD TRIPLES uDOF BY EXPLOITING ARRAY MOTION

Arrays	Static	Method in [33]	Proposed
CPA	15	21	45
NA	23	25	69
ANA	25	27	75
MRA	27	29	81
MISC array	27	29	81

[38]. We assume $v = 2$ m/s in the above example. Thus, the maximal synthesis time is $\tau = (2P+1)d/v = 17.25$ s, which is much shorter than the theoretical coherence time. Hence, the proposed method is feasible in practice.

The idea of the selection of τ is inspired by Cantor array [27], which is defined recursively as the union set of two identical shorter Cantor arrays with the distance of array aperture. However, the differences between our method and the Cantor array method are as follows. First, Cantor array is physically constructed while our synthetic array is synthesized by exploiting array motion. Second, the difference coarray of a Cantor array is hole-free whereas that of the sparse array in the letter does not have to be hole-free. Third, in addition to Cantor arrays, the sparse array in the proposed method can be other linear array structures such as CPAs and NAs.

IV. SIMULATION RESULTS

In this section, numerical examples are given to demonstrate the superiority of the proposed method. We compare the proposed method to the state-of-the-art method in [33] for different arrays with six sensors including a CPA, an NA, an ANA, an MISC array and an MRA, which are given as follows.

CPA:$\{0, 2, 3, 4, 6, 9\}$; NA: $\{0, 1, 2, 3, 7, 11\}$;
ANA: $\{0, 1, 3, 7, 11, 12\}$; MRA:$\{0, 1, 6, 9, 11, 13\}$;
MISC array:$\{0, 1, 2, 6, 10, 13\}$.

First, the uDOF of these arrays for different scenarios are given in Table I, where the second column ("Static") represents the uDOF of physical arrays without exploiting array motion. According to Table I, the proposed method tripes uDOF by employing array motion. In contrast, the method in [33] only increases uDOF by a small number. Especially for NAs, ANAs, MISC arrays and MRAs, the number is only 2.

Then, the spatial spectra of the two methods for different arrays are depicted in Fig. 2 by using coarray MUSIC [36], where we consider that all sparse arrays are moving at speed $v = 2$ m/s, impinged by $K = 9$ narrowband sources with $f = 3$ kHz. The sources are corrupted by Gaussian noise with SNR $= 0$ dB and uniformly distributed over $\bar{\theta} = [-0.2, 0.2]$. We compare the two methods over the same observed period, i.e., $T/(2f_s) + D_{\mathrm{M}}d/v = 3.4$ s, where $T = 400$ is the number of snapshots used in the proposed method, the sampling rate $f_s = 8$ kHz, the uDOF of the MRA $D_{\mathrm{M}} = 27$ and the intersensor spacing $d = \lambda/2 = 0.25$ m. It means that the method in [33] can use 27200 snapshots. It can be seen that the method in [33] shown in blue in Fig. 2 fails to detect all sources for all arrays. In contrast, for all arrays, the proposed method shown in red in Fig. 2 can identify all sources clearly. In one word, the proposed method outperforms the method in [33]. This is because the

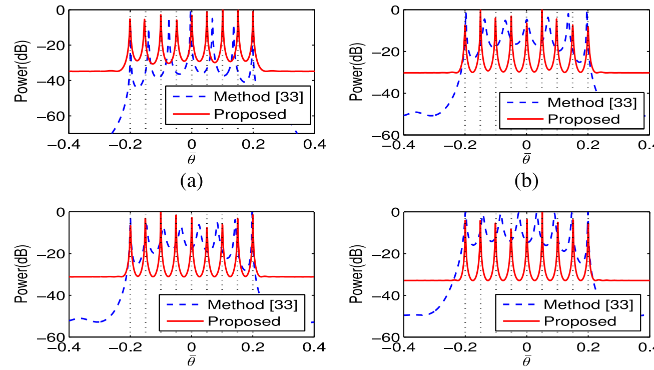

Fig. 2. Spatial spectra of the proposed method and the method in [33] for (a) CPA, (b) NA, (c) ANA, (d) MRA. The dotted vertical lines mark real directions. The method in [33] shown in blue fails to detect all sources for these sparse arrays while the proposed method shown in red can identify all sources clearly.

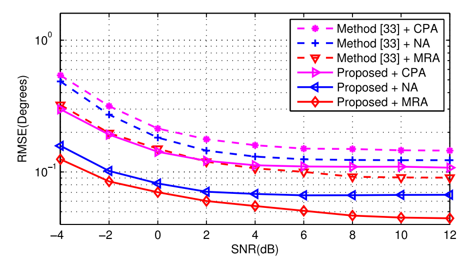

Fig. 3. RMSEs versus SNR for different arrays. For the same array, the proposed method achieves lower errors compared to the method in [33].

proposed method achieves much more uDOF than the method in [33] as illustrated in Table I.

Finally, the root mean square errors (RMSEs) of the proposed method and the method in [33] for different arrays are plotted in Fig. 3 by averaging 300 trials when SNR varies from -4 dB to 12 dB with step 2 dB. All parameters are kept unchanged with the previous one except the source directions $\bar{\theta} = \{-0.4, -0.14, 0.04, 0.1, 0.25, 0.38\}$. It can be seen that for the same array, the proposed method achieves lower errors compared to the method in [33]. For the same method, the least error is achieved by the MRA, followed by the NA and finally the CPA. This is consistent with their uDOF shown in Table I. In general, higher uDOF leads to better performance.

V. CONCLUSION

In this letter, a new method to construct virtual arrays is proposed for moving sparse arrays. First, we prove that the proposed method triples DOF for arbitrary sparse arrays. Thus, the maximal number of detectable sources for DOA estimation is increased threefold. Moreover, we prove that the difference coarray of synthetic arrays is a hole-free ULA if the difference coarray of original arrays is hole-free. Hence, better DOA estimation performance can be obtained for ULA based methods by fully exploiting all sensors in the difference coarray. Simulation results demonstrate that the proposed method can detect more sources and achieve higher resolution than state-of-the-art methods.

REFERENCES

[1] J. Li and P. Stoica, *MIMO Radar Signal Processing*. Hoboken, NJ, USA: Wiley, 2009.

[2] H. Kim and M. Viberg, "Two decades of array signal processing research," *IEEE Signal Process. Mag.*, vol. 13, no. 4, pp. 67–94, Jul. 1996.

[3] J.-A. Luo, X.-H. Shao, D.-L. Peng, and X.-P. Zhang, "A novel subspace approach for bearing-only target localization," *IEEE Sensors J.*, vol. 19, no. 18, pp. 8174–8182, Sep. 2019.

[4] H.-J. Shao, X.-P. Zhang, and Z. Wang, "Efficient closed-form algorithms for AOA based self-localization of sensor nodes using auxiliary variables," *IEEE Trans. Signal Process.*, vol. 62, no. 10, pp. 2580–2594, May 2014.

[5] Z. Wang, J.-A. Luo, and X.-P. Zhang, "A novel location-penalized maximum likelihood estimator for bearing-only target localization," *IEEE Trans. Signal Process.*, vol. 60, no. 12, pp. 6166–6181, Dec. 2012.

[6] S. U. Pillai, Y. Bar-Ness, and F. Haber, "A new approach to array geometry for improved spatial spectrum estimation," *Proc. IEEE*, vol. 73, no. 10, pp. 1522–1524, Oct. 1985.

[7] Y. I. Abramovich, D. A. Gray, A. Y. Gorokhov, and N. K. Spencer, "Positive-definite toeplitz completion in DOA estimation for nonuniform linear antenna arrays. I. Fully augmentable arrays," *IEEE Trans. Signal Process.*, vol. 46, no. 9, pp. 2458–2471, Sep. 1998.

[8] Y. I. Abramovich, N. K. Spencer, and A. Y. Gorokhov, "Positive-definite toeplitz completion in DOA estimation for nonuniform linear antenna arrays. II. partially augmentable arrays," *IEEE Trans. Signal Process.*, vol. 47, no. 6, pp. 1502–1521, Jun. 1999.

[9] M. C. Dogan and J. M. Mendel, "Applications of cumulants to array processing. I. aperture extension and array calibration," *IEEE Trans. Signal Process.*, vol. 43, no. 5, pp. 1200–1216, May 1995.

[10] P. Chevalier and A. Ferreol, "On the virtual array concept for the fourth-order direction finding problem," *IEEE Trans. Signal Process.*, vol. 47, no. 9, pp. 2592–2595, Sep. 1999.

[11] P. Chevalier, A. Ferréol, and L. Albera, "High-resolution direction finding from higher order statistics: The $2q$-MUSIC algorithm," *IEEE Trans. Signal Process.*, vol. 54, no. 8, pp. 2986–2997, Aug. 2006.

[12] W.-K. Ma, T.-H. Hsieh, and C.-Y. Chi, "DOA estimation of quasi-stationary signals with less sensors than sources and unknown spatial noise covariance: A Khatri-Rao subspace approach," *IEEE Trans. Signal Process.*, vol. 58, no. 4, pp. 2168–2180, Apr. 2010.

[13] J. A. Fawcett, "Synthetic aperture processing for a towed array and a moving source," *J. Acoust. Soc. Amer.*, vol. 94, no. 5, pp. 2832–2837, 1993.

[14] S. Kim, D. H. Youn, and C. Lee, "Temporal domain processing for a synthetic aperture array," *IEEE J. Ocean. Eng.*, vol. 27, no. 2, pp. 322–327, Apr. 2002.

[15] S. Stergiopoulos and H. Urban, "A new passive synthetic aperture technique for towed arrays," *IEEE J. Ocean. Eng.*, vol. 17, no. 1, pp. 16–25, Jan. 1992.

[16] A. Moffet, "Minimum-redundancy linear arrays," *IEEE Trans. Antennas Propag.*, vol. AP-16, no. 2, pp. 172–175, Mar. 1968.

[17] P. Pal and P. P. Vaidyanathan, "Nested arrays: A novel approach to array processing with enhanced degrees of freedom," *IEEE Trans. Signal Process.*, vol. 58, no. 8, pp. 4167–4181, Aug. 2010.

[18] P. Pal and P. P. Vaidyanathan, "Coprime sampling and the music algorithm," in *Proc. Digit. Signal Process. Signal Process. Educ. Meeting*, Sedona, AZ, USA, Jan. 2011, pp. 289–294.

[19] P. P. Vaidyanathan and P. Pal, "Sparse sensing with co-prime samplers and arrays," *IEEE Trans. Signal Process.*, vol. 59, no. 2, pp. 573–586, Feb. 2011.

[20] S. Qin, Y. D. Zhang, and M. G. Amin, "Generalized coprime array configurations for direction-of-arrival estimation," *IEEE Trans. Signal Process.*, vol. 63, no. 6, pp. 1377–1390, Mar. 2015.

[21] C. Liu and P. P. Vaidyanathan, "Super nested arrays: Linear sparse arrays with reduced mutual coupling—Part I: Fundamentals," *IEEE Trans. Signal Process.*, vol. 64, no. 15, pp. 3997–4012, Aug. 2016.

[22] C. Liu and P. P. Vaidyanathan, "Super nested arrays: Linear sparse arrays with reduced mutual coupling—Part II: High-order extensions," *IEEE Trans. Signal Process.*, vol. 64, no. 16, pp. 4203–4217, Aug. 2016.

[23] J. Liu, Y. Zhang, Y. Lu, S. Ren, and S. Cao, "Augmented nested arrays with enhanced DOF and reduced mutual coupling," *IEEE Trans. Signal Process.*, vol. 65, no. 21, pp. 5549–5563, Nov. 2017.

[24] M. Yang, L. Sun, X. Yuan, and B. Chen, "Improved nested array with hole-free DCA and more degrees of freedom," *Electron. Lett.*, vol. 52, no. 25, pp. 2068–2070, 2016.

[25] J. Shi, G. Hu, X. Zhang, and H. Zhou, "Generalized nested array: Optimization for degrees of freedom and mutual coupling," *IEEE Commun. Lett.*, vol. 22, no. 6, pp. 1208–1211, Jun. 2018.

[26] Z. Zheng, W.-Q. Wang, Y. Kong, and Y. D. Zhang, "MISC array: A new sparse array design achieving increased degrees of freedom and reduced mutual coupling effect," *IEEE Trans. Signal Process.*, vol. 67, no. 7, pp. 1728–1741, Apr. 2019.

[27] C.-L. Liu and P. Vaidyanathan, "Maximally economic sparse arrays and cantor arrays," in *Proc. IEEE 7th Int. Workshop Comput. Advances Multi-Sensor Adaptive Process.*, 2017, pp. 1–5.

[28] V. Sridhar, M. Willerton, and A. Manikas, "Towed arrays: Channel estimation, tracking and beamforming," in *Beamforming: Sensor Signal Processing for Defence Applications*. Singapore: World Scientific, 2015, pp. 189–219.

[29] J. Ramirez, J. Odom, and J. Krolik, "Exploiting array motion for augmentation of co-prime arrays," in *Proc. IEEE 8th Sensor Array Multichannel Signal Process. Workshop*, A Coruna, Spain, Jun. 2014, pp. 525–528.

[30] J. Ramirez and J. Krolik, "Multiple source localization with moving co-prime arrays," in *Proc. IEEE Int. Conf. Acoust., Speech Signal Process.*, 2015, pp. 2374–2378.

[31] J. Ramirez and J. Krolik, "Synthetic aperture processing for passive co-prime linear sensor arrays," *Digit. Signal Process.*, vol. 61, pp. 62–75, 2017.

[32] G. Qin, M. G. Amin, and Y. D. Zhang, "Analysis of coprime arrays on moving platform," in *Proc. IEEE Int. Conf. Acoust., Speech Signal Process.* Brighton, U.K., 2019, pp. 4205–4209.

[33] G. Qin, M. G. Amin, and Y. D. Zhang, "DOA estimation exploiting sparse array motions," *IEEE Trans. Signal Process.*, vol. 67, no. 11, pp. 3013–3027, Jun. 2019.

[34] G. Qin, Y. D. Zhang, and M. G. Amin, "DOA estimation exploiting moving dilated nested arrays," *IEEE Signal Process. Lett.*, vol. 26, no. 3, pp. 490–494, Mar. 2019.

[35] C.-L. Liu and P. P. Vaidyanathan, "Robustness of difference coarrays of sparse arrays to sensor failures—Part I: A theory motivated by coarray MUSIC," *IEEE Trans. Signal Process.*, vol. 67, no. 12, pp. 3213–3226, Jun. 2019.

[36] C. Liu and P. P. Vaidyanathan, "Remarks on the spatial smoothing step in coarray MUSIC," *IEEE Signal Process. Lett.*, vol. 22, no. 9, pp. 1438–1442, Sep. 2015.

[37] Z. Zheng, Y. Huang, W.-Q. Wang, and H. C. So, "Spatial smoothing PAST algorithm for DOA tracking using difference coarray," *IEEE Signal Process. Lett.*, vol. 26, no. 11, pp. 1623–1627, Nov. 2019.

[38] T. Yang, "Measurements of temporal coherence of sound transmissions through shallow water," *J. Acoust. Soc. Amer.*, vol. 120, no. 5, pp. 2595–2614, 2006.