Increasing the Uniform Degrees of Freedom for Moving \(q\)-Dilated Arrays

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Abstract—For \(q\)-dilated arrays, i.e., dilated arrays with dilation factor \(q\), the uniform degrees of freedom (uDOFs) of the synthetic array after array motion can be increased by a factor of \(q\) compared with that of original linear arrays for \(q \leq 3\). However, when \(q \geq 4\), the number of uDOFs of the synthetic arrays of \(q\)-dilated arrays is only three using existing moving array models. To achieve a high number of uDOFs for \(q\)-dilated arrays with \(q \geq 4\), in this article, we design a new moving array processing model in \(q\)-dilated arrays, which synthesizes multiple shifted arrays with displacements of half wavelength multiples. The number of shifted arrays in the proposed model is shown to be a function of the dilation factor \(q\). First, we prove that the number of uDOFs of synthetic arrays of \(q\)-dilated arrays can be \(q\) times that of their original arrays for arbitrary positive integer \(q\). Hence, the maximum number of detectable sources for direction-of-arrival (DOA) estimation is increased by a factor of \(q\). Second, we apply the new model to two-parallel \(q\)-dilated arrays to estimate the 2-D DOAs, increasing the number of identifiable sources by a factor of \(q\) for 2-D DOA estimation. Numerical examples show the superiority of the proposed model based on \(q\)-dilated arrays.

Index Terms—Degrees of freedom (DOFs), dilated arrays, direction-of-arrival (DOA) estimation, moving arrays, parallel arrays, sparse arrays.

I. INTRODUCTION

TARGET localization or tracking is a main topic in Internet of Things and has been widely used in many applications, such as battlefield surveillance, border patrol, and search and rescue [1], [2], [3], [4], [5], [6], [7]. In many of these applications, the localization or tracking of targets is generally done by a two-stage process. First, the sensor nodes estimate the direction of arrivals (DOAs) or angle of arrivals of the targets. Then, the location or trajectory of the targets is obtained by using data fusion algorithms according to the estimated DOAs. Typically, to perform DOA estimation, the sensor node is equipped with a sensor array, such as an antenna array and an acoustic array. For instance, in acoustic sensor array networks, sensor nodes exploit microphone arrays to compute the DOAs of targets and then to localize and track the targets according to the DOA estimates [4], [5].

In Internet of mobile things, sensor arrays are mounted on moving platforms, such as autonomous underwater vehicles (AUVs), unmanned aerial vehicles, airplanes, and ships. For example, in [8], to cooperatively locate an acoustic source, a team of AUVs is exploited and each of the AUV tows a hydrophone array to estimate the DOA of the source. In [9], an airborne antenna array is employed to investigate the 3-D localization problem for multiple emitters. More examples regarding moving sensor arrays can be found in [10], [11], [12], and [13] for Internet of Things applications.

When sensor arrays move, synthetic aperture (SA) technology [14], [15] can be employed to increase aperture and degrees of freedom (DOFs). Thus, more sources can be detected by using arrays on a moving platform compared to that on a stationary platform.

Over the past decades, uniform linear array (ULA) is one of the most frequently utilized array geometries due to its simplicity and uniformity. An \(N\)-sensor ULA can identify at most \(N–1\) sources by making use of traditional DOA estimation approaches such as multiple signal classification (MUSIC) [16]. When applying SA technology to ULA, the aperture and DOFs can also be increased. However, to increase DOF by \(N\), the array needs to travel \(N\) times half wavelength during the processing time period. This is impractical for a large \(N\).

Recently, most sparse arrays [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], such as nested arrays (NAs) [18], augmented NAs (ANAs) [23], and co-prime arrays (CPAs) [19], [20], have been proposed on a static platform. These arrays are applicable on a moving platform in Internet of mobile things as well. Compared to ULAs, these arrays can achieve much higher DOFs and uniform DOFs (uDOFs), which are required by many high-resolution sensing and imaging methods in applications, such as radar and sonar sensor networks.

In this article, we focus on moving sparse sensor arrays. When a sparse array moves, its DOF and uDOF can be further increased by using SA technology [28], [29], [30], [31], [32].

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[33], [34], [35], [36]. In [28], [29], and [30], SA technology is employed to fill in the holes in the difference coarray of a CPA such that a hole-free ULA is obtained. Assuming that the array is displaced by half a wavelength, the increment of uDOF for certain sparse arrays is derived, including CPAs, NAs, multiple level NAs, and sparse ULA in [31]. Unfortunately, since these approaches focus on filling in holes in the corresponding difference coarray, the increment of uDOF for these sparse arrays is quite limited especially under the restriction of half-wavelength array motion. Particularly, for any array whose difference coarray is already a ULA, the number of uDOFs is only increased by two.

To further increase uDOF, in [32], the interelement spacing of NAs is dilated multifold on a moving platform, yielding a new array named dilated NAs. Dilated NAs can acquire a fully filled (hole-free) difference coarray with at most three times higher uDOFs than NAs. However, first, the dilation method is applied only to NAs since NAs have a relatively simple closed-form expression for their array structure. Second, the number of uDOFs achieved by dilated NAs is only three for dilation factor greater than or equal to four because the moving distance of the array during the processing period is confined to half wavelength in existing models. Under the half-wavelength restriction, the difference coarray of the synthetic array of a dilated NA with dilation factor greater than or equal to 4 has many holes such that the number of uDOFs achieved by dilated NAs is only three. In [33] and [34], q-dilated arrays, i.e., dilated arrays with dilation factor q, have been proposed by generalizing the dilation method from NAs to arbitrary linear arrays. The q-dilated arrays can increase uDOF by a factor of q for q ≤ 3 for any linear arrays. Furthermore, two-parallel dilated arrays have been presented in [34], showing that the maximum number of identifiable sources is increased by a factor of q for q ≤ 3 as well for 2-D DOA estimation. However, the second issue still exists, i.e., the number of uDOFs of the synthetic array is only three for q-dilated arrays with dilation factor q ≥ 4 using existing models.

In this article, to achieve a high number of uDOFs for q-dilated arrays with q ≥ 4, we design a new moving array processing model in q-dilated arrays, which synthesizes multiple shifted arrays with displacements of half wavelength multiples. First, we prove that the numbers of the DOFs and uDOFs of synthetic arrays of q-dilated arrays can be q times those of their original arrays for arbitrary positive integer q. Therefore, through the use of q-dilated arrays, the maximum number of detectable sources is increased q-fold. Furthermore, we apply the new model to two-parallel q-dilated arrays to perform 2-D DOA estimation, increasing the number of identifiable sources by a factor of q for 2-D DOA estimation. The effectiveness of the proposed model is verified by comparing it with existing state-of-the-art models in terms of spatial spectra and root mean-square error (RMSE).

To further clarify the contribution of this article, we compare the differences between this article and our previous paper [34], as follows.

First, we propose a new moving array model, which synthesizes multiple shifted arrays with displacements of half wavelength multiples. The number of shifted arrays in the proposed model is shown to be a function of the dilation factor while in [34], the number of shifted arrays in that model is fixed to one. Second, the numbers of the DOFs and uDOFs of the synthetic array of q-dilated arrays in this article are comprehensively analyzed for arbitrary positive integer number of shifted arrays. We prove that the number of uDOFs of synthetic arrays of q-dilated arrays can be q times that of their original arrays for arbitrary positive integer q. In comparison, the numbers of the DOFs and uDOFs of synthetic arrays of q-dilated arrays in [34] are only analyzed when the number of shifted arrays is fixed to one. It is shown in [34] that the maximum uDOF of synthetic arrays of q-dilated arrays is only three times that of their original array for q = 3.

Also note that in this article, the numbers of the DOFs and uDOFs achieved by q-dilated arrays are shown to be the functions of the dilation factor q, the number of shifted arrays, and the geometry of the original array, all of which require to be carefully set to obtain a higher number of uDOFs. Therefore, this article is not a simple extension of [34].

The remainder of this article organized is as follows. First, we review definitions about sparse arrays and propose a new model of moving array processing for q-dilated arrays in Section II. Based on the new model, we formulate the problem in this article. Then, in Section III, we derive the number of DOFs and uDOFs of synthetic arrays and present important results for q-dilated arrays. Next, in Section IV, we generalize our model into 2-D DOA estimation, showing how to increase the maximum number of identifiable sources for 2-D DOA estimation. After that, in Section V, numerical experiments are provided to verify the effectiveness of the proposed model based on q-dilated arrays. Finally, Section VI concludes this article.

Notations: Lowercase letters, bold lowercase letters, and bold uppercase letters represent scalars, vectors, and matrices, respectively. The symbol (·)T refers to the transpose of a matrix or a vector. The cardinality of a finite set denoted by |N|. For a real number x, |x| rounds x to the largest integer smaller than or equal to x. Given a vector a, diag(a) acquires a diagonal matrix with a being its diagonal. We utilize an integer set to represent the sensor location of an array after normalization by underlying intersensor spacing d. For example, a set N = {n1, . . . , nN} denotes a sensor array with N sensors, whose ith sensor location is nijd.

II. PRELIMINARIES

In this section, necessary definitions about sparse linear arrays are defined first. Then, we formulate the 1-D moving dilated array model and our problem in this article.

A. Definitions

Definition 1 (Difference Coarray) [18], [21]: The difference coarray of a linear array N = {ni|ni, i = 1, . . . , N} is defined as

\[ \mathbb{D}(N) = \{n_i - n_j | n_i, n_j \in N\} \] (1)
Note that in this article $\mathbb{D}(N)$ is treated as a function of $N$. In addition, $|\mathbb{D}(N)| > |N|$. For example, the difference coarray of $N = \{1, 2, 5\}$ is $\mathbb{D}(N) = \{0, \pm 1, \pm 3, \pm 4\}$. Therefore, $|\mathbb{D}(N)| = 7$ is more than $|N| = 3$.

**Definition 2:** Given an integer $z$ and two integer sets $N = \{n_i|n_i, i = 1, \ldots, N\}$ and $M = \{m_j|m_j, i = 1, \ldots, M\}$, some set operations are defined as follows [23].

**Translation Set:** $N + z = \{n_i + z|i = 1, \ldots, N\}$

**Dilation Set:** $\mathbb{C}(N, M) = \{n_i - m_j|i = 1, \ldots, N, j = 1, \ldots, M\}$.

It can be derived that $\mathbb{D}(N + z) = \mathbb{D}(N)$, $\mathbb{D}(zN) = z\mathbb{D}(N)$, $\mathbb{C}(N, M) = \mathbb{D}(N, M)$, $\mathbb{C}(N + z, N) = \mathbb{D}(N) + z$ and

\[
\mathbb{D} \left( \bigcup_{i=1}^{z} N_i \right) = \left[ \bigcup_{i=1}^{z} \mathbb{D}(N_i) \right] \bigcup \left[ \bigcup_{i=1}^{z} \mathbb{C}(N_i, N_j) \right]
\]

where $z$ and $Z$ are two integers. Note that $\mathbb{C}(N, M) = \mathbb{C}(M, N) \neq \mathbb{C}(M, N)$ if $N \neq M$.

**Definition 3 (DOFs and uDOFs) [21]:** For a given linear array $N$, assume that $\mathbb{D}(N)$ and $\mathbb{U}(N)$ are its difference coarray and the longest central ULA in its difference coarray, respectively. The cardinality of $\mathbb{D}(N)$ and $\mathbb{U}(N)$, i.e., $|\mathbb{D}(N)|$ and $|\mathbb{U}(N)|$, is called the numbers of DOFs and uDOFs of the linear array, respectively.

For DOA estimation, the maximal number of detectable sources is decided by the number of DOFs and uDOFs. For instance, the maximal number of detectable sources is $(|\mathbb{U}(N)| - 1)/2$ when coarray MUSIC [18], [37] is applied to a sparse array $N$ for DOA estimation.

**Definition 4 (Island [34]):** For a linear array $N$, an island in $N$ is defined as a subset of $N$ that is composed of the longest consecutive sensor segment including a certain sensor.

For example, for $N = \{1, 2, 6, 10\}$, it has three islands, i.e., $\{1, 2\}$, $\{6\}$, and $\{10\}$. The number of islands $N$ is 3. Note that in $N$, neither $\{1\}$ nor $\{2\}$ is an island. Particularly, a ULA has only one island. Given a linear array $N$ with $N$ elements, let the number of islands in $N$ be $I$ and the number of elements in the $i$th island be $I_i$. Then, it can be derived that $\sum_{i=1}^{I} I_i = N$.

**Definition 5 (Dilated Arrays [33], [34]):** The dilated arrays are constructed by enlarging the interelement spacing of arbitrary linear arrays by a factor of $q$, where $q$ is a positive integer, i.e., given a linear array $L = \{i_l|l, i = 1, \ldots, N\}$

its $q$-dilated array (dilated array with dilation factor $q$) $N_q$ is constructed as

\[N_q = qL.\]

The sparse array $L$, known as the original array of the $q$-dilated arrays, can be arbitrary linear array geometry, such as an NA and a CPA. The resulting $q$-dilated arrays are named $q$-dilated CPA ($q$-DCPA) and $q$-dilated NA ($q$-DNA). It has been shown that $q$-dilated arrays ($q \leq 3$) can achieve $q$ times the number of uDOFs on a moving platform compared to their original arrays [33], [34].

**B. 1-D Moving Dilated Array Model**

Consider that $K$ far-field narrowband sources $\{s_k(t), \ldots, s_K(t)\}$ impinge on a $q$-dilated array $N_q = qL$, whose original array is $L = \{l_1, \ldots, l_L\}$, as depicted in Fig. 1. We make several common assumptions about the sensor array and the sources in the following.

1) The $q$-dilated array is moving along a straight-line course at a constant velocity $v$, which can be estimated in real time. During a short time period, it is reasonable to treat the velocity as a constant. Note that the movement direction of the sensor array can be calibrated in advance to avoid movement deviation. In summary, the assumption is reasonable.

2) The directions of the $K$ sources, i.e., $\theta = [\theta_1, \ldots, \theta_K]^T$, are unchanged during the observed time period.

3) The sources are assumed to be mutually uncorrelated, and statistically independent of the observed noise. The same assumption is employed in a great number of articles in the difference coarray field, such as [18], [19], [20], [21], [22], [23], [24], [25], [28], [29], [30], [31], [32], [33], [34], [35], and [38]. Note that in the multipath scenario the sources may be correlated or coherent. In this case, the difference coarray idea cannot be exploited through the vectorization of the covariance matrix. To the best of our knowledge, there is no good method to obtain the difference coarray of a sparse array when the sources impinged on the sparse array are correlated or coherent. Thus, we do not consider the multipath scenario.

Let $\hat{\theta}_i = d \sin(\theta_i)/\lambda$ denote the normalized DOA of the $k$th source, where $d$ represents the underlying interelement spacing and $\lambda$ is the wavelength of the sources. At time $t$, the measured signal of the $q$-dilated array is [31], [33]

\[x_0(t) = A_0 s(t) + \epsilon_0(t)\]

where $x_0(t)$ and $\epsilon_0(t)$ are received signal and noise vectors with size $N \times 1$, $A_0 = [a(\theta_1), \ldots, a(\theta_K)]$ represents the array manifold with $a(\theta_i) = [\exp(-j2\pi q_1\theta_i), \ldots, \exp(-j2\pi q_m\theta_i)]^T$, and $s(t) = [s_1(t) \exp(-j2\pi \theta_1 v t/d), \ldots, s_K(t) \exp(-j2\pi \theta_K v t/d)]^T$ denotes the source vector at time $t$, respectively.

In the moving array model in [33] and [34], only one shifted array with a displacement of half wavelength is synthesized. Here, different from that in [33] and [34], we synthesize $R$ shifted arrays with displacements of half wavelength multiples in the following.
Let arrays $R$ can be acquired in the Khatri–Rao subspace through the synthetic array, respectively. 

![Fig. 2. Synthetic arrays of dilated CPAs](left) and their difference coarrays $\mathbb{D}(S_q^R)$ (right) for different dilation factors $q$ and different numbers of shifted arrays $R$. (a) Synthetic array, $q = 1, R = 0$, (b) difference coarray, $q = 1, R = 0$, (c) synthetic array, $q = 3, R = 1$, (d) difference coarray, $q = 3, R = 1$, (e) synthetic array, $q = 4, R = 1$, (f) difference coarray, $q = 4, R = 1$, (g) synthetic array, $q = 4, R = 2$, (h) difference coarray, $q = 4, R = 2$, (i) synthetic array, $q = 5, R = 2$, and (j) difference coarray, $q = 5, R = 2$. Black, yellow, and blue circles denote physical sensors, shifted sensors, and sensors from difference coarray, respectively. The numbers of the uDOFs of the five synthetic arrays are 15, 45, 3, 61, and 75, respectively. It implies that the uDOFs of the synthetic arrays are increased by a factor of $q$ compared to that of the CPA if $R = \lfloor q/2 \rfloor$. In contrast, if $R < \lfloor q/2 \rfloor$, the number of uDOFs of the synthetic arrays degrades significantly and is only $2R + 1$.

At time $t + rt$, where $r = 1, \ldots, R$ and $\tau = d/v$ refers to the time needed for the $q$-dilated array to move a distance of $d$, the measured signal becomes

$$x_r(t + rt) = A_0s(t + rt) + \epsilon_0(t + rt).$$

(6)

For narrowband sources [28], [29], [30], [31], [32], [33], [34], [35], [36], we have

$$s_k(t + rt) \approx \exp(j2\pi f rt)s_k(t)$$

(7)

where $f$ stands for the central frequency of the sources. Let $x_r(t) = x_0(t + rt) \exp(-j2\pi fr t)$, $\epsilon_r(t) = \epsilon_0(t + rt) \exp(-j2\pi fr t)$ and $A_r = [a_r(\bar{\theta}_0), \ldots, a_r(\bar{\theta}_K)]$ with

$$a_r(\bar{\theta}_k) = a(\bar{\theta}_k) \exp(-j2\pi fr_k).$$

(8)

Then, we arrive at

$$x_r(t) = A_r s(t) + \epsilon_r(t), \quad r = 1, \ldots, R.$$ 

(9)

Compared (9) to (5), $x_r(t)$ can be considered as the measured signal of a virtual array, which is a shifted array of $\mathbb{N}_q^0$ displaced by $rt = rd$. By stacking all $x_r(t)$ for $r = 0, \ldots, R$ in (9), we obtain a synthetic array output

$$y(t) = Bs(t) + \eta(t)$$

(10)

where $B = [A_0^T, \ldots, A_R^T]^T$ and $\eta(t) = [\epsilon_0^T(t), \ldots, \epsilon_R^T]^T$ represent the array manifold and the received noise vector of the synthetic array, respectively.

Eventually, the difference coarray of this synthetic array can be acquired in the Khatri–Rao subspace through the vectorization of the covariance matrix of $y(t)$ [18], [39].

Note that similar to the moving array model in [29], the proposed moving array model also synthesizes multiple shifted arrays. However, the number of shifted arrays in the two models is completely different. In Section III, we will show that for the proposed model, it is suggested that the number of shifted arrays $R = \lfloor q/2 \rfloor$. In comparison, the number of shifted arrays in the model in [29] is only related to the co-prime pair for a given CPA. This difference comes from the different objectives of the two papers. Specifically, in this article, we aim to obtain a large ULA segment in the difference coarray of the synthetic array of a dilated array. It implies that the obtained difference coarray can have holes. Ramirez and Krolik [29] expected to obtain a hole-free difference coarray only for a given CPA.

C. Problem Formulation

Based on existing models, dilated arrays can increase DOF and uDOF by a factor of $q$ for $q \leq 3$ in comparison to their original arrays. However, in the dilated arrays [33], [34], the number of shifted arrays is fixed as $R = 1$. Unfortunately, when $R = 1$ and $q \geq 4$, the difference coarray of the corresponding synthetic array has many holes such that the number of uDOFs of the synthetic array of $q$-dilated arrays decreases to three. As demonstrated in Fig. 2, we take a 3-dilated CPA $\{0, 6, 9, 12, 18, 27\}$ as an example. Its original array is $\{0, 2, 3, 4, 6, 9\}$, shown in Fig. 2(a). In Fig. 2(c), the corresponding synthetic array of the 3-dilated array for $R = 1$ is plotted. From the difference coarrays of the original array and the synthetic array, depicted in Fig. 2(b) and (d), respectively, it can be clearly noted that the number of uDOFs of the synthetic array is 45, exactly that of the original array multiplied by three. However, when $q$ increases, the difference coarray of the synthetic array has many holes. As illustrated in Fig. 2(e) and (f), where $q = 4$, the number of uDOFs achieved by the dilated arrays is only three.

According to (8), the shifted array of the $q$-dilated array displaced by $rd$ can be given by

$$\mathbb{N}_q^r = \mathbb{N}_q^0 + r, \quad r = 1, \ldots, R.$$ 

(11)

Then, we can obtain the synthetic array as

$$S_q^R = \mathbb{N}_q^0 \cup \mathbb{N}_q^1 \cup \cdots \cup \mathbb{N}_q^R.$$ 

(12)

Given arbitrary positive integer $q$ as well as arbitrary original linear array $L$, in this article, our objective is to find a way to achieve a high number of uDOFs for $q$-dilated arrays including $q \geq 4$, i.e., to obtain a large $|\mathbb{U}(S_q^R)|$.

III. DOF ANALYSIS FOR q-DILATED ARRAYS

To solve the issue in Section II-C, the number of shifted arrays $R$ cannot be fixed as one. In the following, we analyze the numbers of the DOFs and uDOFs of synthetic arrays for different $R$ and $q$ and present the important results about DOF and uDOF.
First, according to (11), it is not difficult to obtain
\[ D\left(N^r_q\right) = D\left(N^0_q\right), \quad r = 1, \ldots, R \]
which indicates that a q-dilated array and its shifted arrays with displacement rd share the same difference coarray.

Then, we obtain Lemma 1, as follows.

**Lemma 1.** For two integers i and r, we have
\[ \sum_{r=-R}^{R} qD(L) + r = \frac{R}{q} \]

**Proof:** See Appendix A.

Lemma 1 is used to prove Theorem 1.

**Theorem 1:** For a q-dilated array \( N^0_q = qL \), where \( L \) denotes its original array. The difference coarray of \( S^R_q \) in (12), i.e., \( D(S^R_q) \), can be obtained by
\[ D\left(S^R_q\right) = \bigcup_{r=-R}^{R} qD(L) + r. \]

**Proof:** See Appendix B.

It is known from Theorem 1 that the difference coarray of the synthetic array can be acquired from that of the original array in three steps. First, dilate the intersensor spacing of the difference coarray of the original array by a factor of q to obtain the diluted difference coarray, denoted by the set \( qD(L) \); second, for \( r = -R, -R+1, \ldots, R \), obtain the shifted versions of the diluted difference coarray with rd, denoted by \( qD(L) + r \); and finally, compute the union of all the \( 2R + 1 \) sets denoting shifted arrays, i.e., \( \bigcup_{r=-R}^{R} qD(L) + r \).

**Theorem 2 (DOF and uDOF) Given a q-dilated array \( N^0_q = qL \) with original array \( L \), and the corresponding synthetic array \( S^R_q \) in (12), the relationship between \( |D(S^R_q)| \) and \( |D(L)| \), and that between \( |U(S^R_q)| \) and \( |U(L)| \) are shown in Table I.

**Proof:** See Appendix C.

**Remark 1:** Theorem 2 holds for any positive integers R and q including the case of \( R = 1 \), which is addressed in [34]. It indicates that the result about the DOFs and uDOFs achieved by q-dilated arrays in [34] is a special case of Theorem 2 with \( R = 1 \).

**Remark 2:** The synthesis time should be shorter than the signal coherence time, which means the time duration over which the signal is coherent. The synthesis time includes two parts, i.e., the time of array motion and the time of sampling. The time of sampling is related to the sampling frequency and the number of snapshots while the time of array motion is \( \tau \). Note that over the synthesis time period, DOAs are assumed to be unchanged. Therefore, the number of shifted arrays \( R \) should be relatively small. Fortunately, when \( q \) is small, the numbers of the DOFs and uDOFs achieved by q-dilated arrays can be increased substantially with a small \( R \). For instance, the numbers of the DOFs and uDOFs for 5-dilated arrays can be increased fivefold when \( R = 2 \), as shown in Table I.

A problem, here, is how to choose \( R \) for a given q-dilated array. According to Remark 2, a smaller \( R \) is preferred under the synthesis time constraint and the assumptions of the array model. However, a smaller \( R \) results in decreased uDOFs. In the following, we analyze the number of uDOFs of the synthetic array to choose \( R \) for a given q-dilated array.

Table I gives the number of uDOFs achieved by q-dilated arrays with different \( R \) and q. In case 1, when \( R < (q-1)/2 \), the number of uDOFs of the synthetic array is 2R + 1, which only depends on the number of shifted arrays \( R \). Therefore, the number of uDOFs of the synthetic array is quite small for \( R < (q-1)/2 \) since \( R \) is small.

Cases 2 and 3 can be regarded as \( R = \lfloor q/2 \rfloor \). In the two cases, we obtain a relatively higher number of uDOFs for q-dilated arrays, which is far more than 2R + 1, the number of uDOFs of case 1 for \( R = \lfloor q/2 \rfloor \). It implies that when \( R \) increases to \( R = \lfloor q/2 \rfloor \), plenty of holes in the difference coarray of the synthetic array are filled. Therefore, the number of uDOFs achieved by q-dilated arrays is significantly increased.

In case 4, when \( R > q/2 \), the number of uDOFs of the synthetic array is increased slowly as \( R \) increases since most holes in the difference coarray of the synthetic array have already been filled when \( R = \lfloor q/2 \rfloor \).

According to the analysis above, when \( R \geq \lfloor q/2 \rfloor \), the number of uDOFs achieved by q-dilated arrays is large and increases slowly with the increase of \( R \). Therefore, in the range \( R \geq \lfloor q/2 \rfloor \), we select a minimum \( R \), that is
\[ R = \lfloor q/2 \rfloor \]

it makes q-dilated arrays achieving a large number of uDOFs with a relatively small \( R \).

We give an example in Fig. 3 to show the number of uDOFs versus \( R \) for q-DNA and q-DCPA with \( q = 4 \) and \( q = 5 \). It can be clearly noted that the results in Fig. 3 are consistent with our analysis. First, when \( R < 2 \), the numbers of the uDOFs for all q-dilated arrays are super small. Second, when \( R \) goes to \( \lfloor q/2 \rfloor = 2 \), the uDOFs for all arrays increase significantly. Third, the uDOFs for all q-dilated arrays increase slowly when \( R > 2 \). It is noted that \( R = \lfloor q/2 \rfloor = 2 \) is a good selection, which is small and makes all the q-dilated arrays achieving a large number of uDOFs.

**Corollary 1:** When \( R = \lfloor q/2 \rfloor \), the number of uDOFs that q-dilated arrays achieve is
\[ \left| U\left(S^R_q\right) \right| = \begin{cases} q|U(L)|, & \text{if } q \text{ is odd} \\ q|U(L)| + 1, & \text{if } q \text{ is even.} \end{cases} \]

**Proof:** Corollary 1 can be directly obtained from Table I. Thus, Corollary 1 is proved.

### Table I: Relationship of uDOF and DOF Between a q-Dilated Array and Its Original Array

| Cases | q     | R            | \( |U(S^R_q)| \) | \( |D(S^R_q)| \) |
|-------|-------|--------------|-----------------|-----------------|
| case 1 | all   | \( R < (q-1)/2 \) | \( 2R + 1 \)     | \( (2R + 1)|D(L)| \) |
| case 2 | even  | \( R = q/2 \)  | \( q|U(L)| \)     | \( q|D(L)| + 1 \) |
| case 3 | odd   | \( R = (q-1)/2 \) | \( q|U(L)| \)     | \( q|D(L)| \) |
| case 4 | all   | \( R > q/2 \)  | \( q|U(L)| + 2R - q + 1 \) | \( q|D(L)| + 2R - q + 1 \) |
Corollary 1 implies that the number of uDOFs achieved by \( q \)-dilated arrays is increased \( q \)-fold compared with that of their original arrays when \( R = \lfloor q/2 \rfloor \). It can also be noted that given a fixed \( R \), a higher number of uDOFs can be achieved for odd \( q = 2R + 1 \) than even \( q = 2R \).

**Corollary 2:** When \( R = \lfloor q/2 \rfloor \), the number of DOFs that \( q \)-dilated arrays achieve is
\[
\|D(qR_q^L)\| = \begin{cases} q\|D(L)\|, & q \text{ is odd} \\ q\|D(L)\| + I_D, & q \text{ is even} \end{cases}
\]
where \( I_D \) denotes the number of islands in \( D(L) \).

**Proof:** See Appendix D.

It can be found that for \( R = \lfloor q/2 \rfloor \), the number of DOFs that a \( q \)-dilated array achieves is at least \( q \) times that of its original array.

An example is given in Fig. 2 to demonstrate the increment of DOF and uDOFs achieved by \( q \)-dilated arrays. In Fig. 2, the original array \( L = \{0, 2, 3, 4, 6, 9\} \), which can be regarded as the synthetic array of its \( q \)-dilated array with \( q = 1 \) and \( R = 0 \), is a CPA with six elements, depicted in Fig. 2(a). The difference coarray of \( L \) is given in Fig. 2(b). It can be seen that the numbers of the DOFs and uDOFs of \( L \) are \( \|D(L)\| = 17 \) and \( |U(L)| = 15 \), respectively. The synthetic arrays for different dilation factor \( q \) and different numbers of shifted arrays are given in the left column of Fig. 2, where \( q = 3 \) and \( R = 1 \) in Fig. 2(c), \( q = 4 \) and \( R = 1 \) in Fig. 2(e), \( q = 4 \) and \( R = 2 \) in Fig. 2(g), and \( q = 5 \) and \( R = 2 \) in Fig. 2(i), respectively. The difference coarrays of these synthetic arrays are plotted in Fig. 2(d), (f), (h), and (j), respectively.

It can be noted that the numbers of the uDOFs of the five synthetic arrays from top to bottom in Fig. 2 are 15, 45, 3, 61, and 75, respectively. Note that \( |U(L)| = 15 \). It implies that the numbers of the uDOFs achieved by \( q \)-dilated arrays are increased by a factor of \( q \) in comparison to that of the original array if \( R = \lfloor q/2 \rfloor \), as demonstrated in Fig. 2(b), (d), (h), and (j), where the dilation factors \( q \) are 1, 3, 4, and 5 and the numbers of the uDOFs of the synthetic arrays are 15, 45, 61, and 75, respectively. The results of the uDOF are in accordance with Corollary 1. In contrast, if \( R < \lfloor q/2 \rfloor \), the number of uDOFs of the synthetic arrays degrades significantly and is only \( 2R + 1 \), as shown in Fig. 2(f), where \( q = 4 \) and \( R = 1 \) and the number of uDOFs of the synthetic arrays is \( 2R + 1 = 3 \). This is why the dilated arrays [33], [34] can only triple the number of uDOFs for \( q = 3 \) under a half-wavelength array motion, i.e., \( R = 1 \). Because when \( R = 1 \), to obtain a long ULA segment in \( D(S_q^R) \), the largest dilation factor is three. As shown in Fig. 2(f), where \( R = 1 \) and \( q = 4 \), there are many holes in the difference coarray.

Meanwhile, it is observed that the numbers of the DOFs of the five synthetic arrays from top to bottom in Fig. 2 are 17, 51, 51, 71, and 85, respectively. According to Corollary 2, for odd \( q \) and \( R = (q - 1)/2 \), the number of DOFs is increased by a factor of \( q \), i.e., \( \|D(S_q^R)\| = \|D(L)\| = 17q \), as shown in Fig. 2(d), where \( q = 3 \), \( R = 1 \) and the number of DOFs is 51, and shown in Fig. 2(j), where \( q = 5 \), \( R = 2 \) and the number of DOFs is 85, respectively.

**Theorem 3:** For a \( q \)-dilated array \( N_q^L = qL \), where \( L \) is its original array. If \( R \geq \lfloor q/2 \rfloor \), and the difference coarray of \( L \), i.e., \( D(L) \), is hole-free, then the difference coarray of the synthetic array, i.e., \( D(S_q^R) \), is a hole-free ULA with \( \|D(S_q^R)\| = q\|D(L)\| + 2R - q + 1 \) elements.

**Proof:** See Appendix E.

Take dilated NAs as an example. The corresponding synthetic arrays and their difference coarrays for different dilation factors and different numbers of shifted arrays are depicted in Fig. 4. The original array is \( L = \{0, 1, 2, 5\} \), shown in Fig. 4(a). The number of DOFs of the original array is \( D(L) = 11 \), as shown in Fig. 4(b). The synthetic arrays with dilation factor \( q = 4 \) and the number of shifted arrays \( R = 1, 2, 3 \) are plotted in Fig. 4(c), (e), and (g), respectively. Their corresponding difference coarrays are depicted in Fig. 4(d), (f), and (h), respectively. As illustrated in Fig. 4(b), (f), and (h), where \( R \geq \lfloor q/2 \rfloor \), the difference coarrays of the synthetic arrays are fully filled ULA with elements 11, 45, and 47, respectively, exactly equal to \( q\|D(L)\| + 2R - q + 1 \).

**Theorem 3** provides a way to acquire a hole-free difference coarray for \( q \)-dilated arrays. It is not hard to see that [21] the hole-free difference coarray property makes it convenient to apply typical ULA-based DOA estimation approaches such as coarray MUSIC [18], [37]. According to Theorem 3, if one expects to acquire a fully filled ULA for the difference coarray of the synthetic array, what one needs to do is to select an original array that has a fully filled difference coarray and to make sure \( R \geq \lfloor q/2 \rfloor \).

**IV. DOF FOR 2-D q-DILATED ARRAYS**

Many 2-D array structures can be leveraged to estimate 2-D DOAs. Among these array structures, parallel array is one of
We define the 2-D normalized DOA as 

\[ \alpha_D = \text{linear arrays with size } N \times v \lambda \]

only three. In this section, first, we present a new moving par-

DOA estimation by a factor of \( q \) when \( q \geq 2 \), and shown in (h), where \( q = 4 \) and \( R = 3 \).

However, \( q \) is hole-free and \( R \geq |q/2| \), as shown in (f), where \( q = 4 \) and \( R = 2 \), and shown in (h), where \( q = 4 \) and \( R = 3 \).

In [34], it has been shown that exploiting a two-parallel dilated array with \( \alpha_1 = [1, \exp(-j2\pi m\alpha_1), \ldots, \exp(-j2\pi n\alpha_1)]^T \) and \( \Phi = \text{diag}((\exp(-j2\pi m\beta_1), \ldots, \exp(-j2\pi m\beta_N))^T) \).

At time \( t+r \tau \), where \( r = 1, \ldots, R \), the array output becomes

\[
\begin{align*}
\mathbf{x}_{0,1}(t+r) &= \mathbf{A}_{0,1}\mathbf{s}(t+r) + \mathbf{e}_{0,1}(t+r) \\
\mathbf{x}_{0,2}(t+r) &= \mathbf{A}_{0,2}\mathbf{s}(t+r) + \mathbf{e}_{0,2}(t+r)
\end{align*}
\]

(20)

Similarly, we can obtain

\[
\begin{align*}
\mathbf{x}_{r,1}(t) &= \mathbf{A}_{r,1}\mathbf{s}(t) + \mathbf{e}_{r,1}(t) \\
\mathbf{x}_{r,2}(t) &= \mathbf{A}_{r,2}\mathbf{s}(t) + \mathbf{e}_{r,2}(t)
\end{align*}
\]

(21)

where \( \mathbf{x}_{r,i}(t) = \mathbf{x}_{0,i}(t+r)\exp(-j2\pi r\tau) \) for \( i = 1, 2 \),

\( \mathbf{e}_{r,i}(t) = \mathbf{e}_{0,i}(t+r)\exp(-j2\pi r\tau) \) for \( i = 1, 2 \),

\( \mathbf{A}_{r,1} = [a_{r,1}(\alpha_1), \ldots, a_{r,1}(\alpha_k)] \) with

\[
\begin{align*}
a_{r,1}(\alpha_k) &= a_{0,1}(\alpha_k) \exp(-j2\pi r\alpha_k)
\end{align*}
\]

(22)

and \( \mathbf{A}_{r,2} = \mathbf{A}_{r,1}\Phi \), respectively.

It can be noted in (21) that a shifted array of the physical parallel array with \( rd \) displacement is acquired, as depicted by the yellow circles in Fig. 5. Combining the physical array and its all shifted ones, we obtain a synthetic array, whose output is

\[
\begin{align*}
\mathbf{y}_1(t) &= \mathbf{B}_1\mathbf{s}(t) + \eta_1(t) \\
\mathbf{y}_2(t) &= \mathbf{B}_2\mathbf{s}(t) + \eta_2(t)
\end{align*}
\]

(23)

where \( \mathbf{y}_1(t) = [\mathbf{x}_{0,1}(t)^T, \ldots, \mathbf{x}_{R,1}(t)^T]^T \), \( \mathbf{y}_2(t) = [\mathbf{x}_{0,2}(t)^T, \ldots, \mathbf{x}_{R,2}(t)^T]^T \), \( \mathbf{B}_1 = [\mathbf{A}_{0,1}^T, \ldots, \mathbf{A}_{R,1}^T]^T \), \( \mathbf{B}_2 = \mathbf{A}_{0,2}\Phi \), \( \eta_1(t) = [\epsilon_{1,0}^T, \ldots, \epsilon_{R,1}^T]^T \), and \( \eta_2(t) = [\epsilon_{1,2}^T, \ldots, \epsilon_{R,2}^T]^T \).

Note that in (23), the obtained synthetic planar array consists of two identical synthetic sparse linear arrays, i.e., \( \mathbb{S}^2 \), given in (12) in Section II-C.

Similar to [34], a virtual parallel array consisting of two longer ULAs can be obtained by constructing a virtual covariance matrix from the covariance and cross-covariance matrices of \( \mathbf{y}_1(t) \) and \( \mathbf{y}_2(t) \). Note that each ULA in the virtual array is exactly the nonnegative part of \( \mathbb{U}(\mathbb{S}^2) \). Then, 2-D DOAs can be computed based on the virtual parallel array exploiting typical 2-D DOA estimation methods, such as polynomial root-finding methods [40] and 2-D MUSIC.
TABLE II
MAXIMUM NUMBER OF RESOLVABLE SOURCES FOR TWO-PARALLEL
q-DILATED ARRAYS WITH 2N ELEMENTS

<table>
<thead>
<tr>
<th>Cases</th>
<th>q</th>
<th>R</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>case 1</td>
<td>all</td>
<td>$R &lt; (q - 1)/2$</td>
<td>$2R$</td>
</tr>
<tr>
<td>case 2</td>
<td>even</td>
<td>$R = q/2$</td>
<td>$q</td>
</tr>
<tr>
<td>case 3</td>
<td>odd</td>
<td>$R = (q - 1)/2$</td>
<td>$q</td>
</tr>
<tr>
<td>case 4</td>
<td>all</td>
<td>$R &gt; q/2$</td>
<td>$\geq q</td>
</tr>
</tbody>
</table>

B. Number of Resolvable Sources

For a parallel array with two ULAs, different 2-D DOA estimation methods can detect different numbers of sources. For example, the 2-D MUSIC method can detect at most $2N - 1$ sources while the polynomial root-finding method [34], [40], [45] can identify at most $2N - 2$ sources with $2N$ elements. However, the 2-D MUSIC method needs 2-D search to find the DOAs, resulting in high computational cost. Here, we utilize the polynomial root-finding method, which transforms 2-D DOA estimation issue into two 1-D ones and can automatically pair the azimuth and elevation angles.

Given a two-parallel array making up of two ULAs with each ULA $N$ elements, the maximum number of resolvable sources for 2-D DOA estimation and is $2(N - 1)$ according to the polynomial root-finding method. Thus, using the proposed synthesis model, the two-parallel $q$-dilated array with $2N$ elements can process at most $|U(S_q^R)| - 1$ sources, where $|U(S_q^R)|$ is given in Table I. Specifically, for a two-parallel $q$-dilated array with $2N$ elements, the maximum number of resolvable sources $D$ is given in Table II.

From Table II, we can see that using a two-parallel $q$-dilated array the maximum number of resolvable sources can be increased by a factor of $q$ in comparison to that using its two-parallel original array for arbitrary positive integer $q$.

V. SIMULATION RESULTS

In this section, experiments are provided to illustrate the superior performance of the proposed methods. First, we compare the presented 1-D DOA estimation method ($q$-dilated arrays with $q = 5$) to the state-of-the-art ones in [34] (3-dilated arrays) and [31] (their original arrays). For 1-D DOA estimation, coarray MUSIC [37] is exploited to obtain the DOAs. We set $R = \lfloor q/2 \rfloor$ for $q$-dilated arrays with $q \geq 4$ according to (16). We select three types of original arrays, including a CPA, an NA, and an ANA. The number of elements is fixed as $N = 8$ for all sparse linear arrays. In the CPA, the coprime integers are given as 2 and 5. In the NA, we set the number of sensors in the subarray of each level as 4 to obtain the largest DOF. For the ANA, we use the first kind of four-level ANA, i.e., ANAII-1. Therefore, the three original arrays are \{0, 2, 4, 5, 6, 8, 10, 15\} for the CPA, \{0, 1, 2, 3, 4, 9, 14, 19\} for the NA, and \{0, 1, 5, 9, 16, 19, 21, 22\} for the ANA.

The corresponding $q$-dilated arrays are $q$-DCPA, $q$-DNA, and $q$-DANA with $q = 3$ and $q = 5$, respectively. The underlying intersensor spacing is $d = \lambda/2$.

Then, exploiting the above sparse linear arrays, we construct their corresponding planar versions with two identical parallel arrays to compare the proposed 2-D DOA estimation approach (two-parallel 5-dilated arrays) to the ones in [34] (two-parallel 3-dilated arrays) and [45] (two-parallel original arrays) for 2-D DOA estimation. Note that here the number of elements is 16 for all parallel arrays. In all parallel arrays, the distance between the two linear subarrays is $\lambda/2$. In these 2-D DOA estimation methods, the polynomial root-find algorithm [40] is applied to estimate the 2-D DOAs.

A. uDOF

Here, we show the numbers of the uDOFs that the three methods achieve for different array structures in Table III. Note that $|U(L)|$ represents the number of uDOFs of original arrays. It can be seen in Table III that the proposed method obtains the highest uDOF among the three methods. Meanwhile, it can be noted that the results in Table III are consistent with Theorem 2. The number of uDOFs that $q$-dilated arrays achieve can be increased by a factor of $q$ using our model. Specifically, the number of uDOFs achieved by $5$-dilated arrays is five times those of their original arrays, where the number of shifted arrays $R = 2$. It means that the number of uDOFs achieved by $5$-dilated arrays is fivefold for only a one-wavelength array motion. Note that for 2-D DOA estimation, the corresponding two-parallel $q$-dilated arrays show similar result about the maximum number of detectable sources, i.e., the maximum number of detectable sources can be increased by a factor of $q$.

It has been shown in [36] that the synthetic array has a lower DOA estimation error compared to its physical array (static array) if the number of uDOFs of the synthetic array divided by the number $\sqrt{2}/2 \approx 1.68$ is greater than that of the physical array. It indicates that the proposed method can have a smaller error theoretically compared with the method in [31]. On the contrary, when dilation factor $q$ is greater than three, the number of uDOFs that the method in [34] achieves is only three for both the CPA and NA due to the constraint of half-wavelength array motion.

B. 1-D Spatial Spectra

Consider that the normalized DOAs of eight closely spaced sources are uniformly distributed over $[-0.045, 0.045]$, corresponding to DOAs in $[-2.58^\circ, 2.58^\circ]$. All sources impinge on the above-mentioned sparse linear arrays, corrupted by additive Gaussian noise with SNR = 5 dB. We plot the spatial spectra of the three 1-D DOA estimation methods for different arrays in Fig. 6 with $T = 1500$ snapshots. In Fig. 6, real directions are marked by vertical lines. A “Failed” text on the
top-right corner in subfigures denotes that the spatial spectrum fails to identify all sources. It can be clearly found that the proposed method can identify all sources by using the NA or the ANA as the original array while the method in [34] can identify all sources only when the original array is the ANA. On the contrary, the spatial spectra of the methods in [31] show missing peaks and fail to identify all sources for all arrays. Meanwhile, for the original array CPA, all three methods fail and have missing peaks. However, the proposed method has fewer missing peaks. In summary, the spatial spectra of the proposed method are more accurate in comparison to those of the methods in [31] and [34].

Then, we give spatial spectra of the proposed method for different dilated arrays to show its superiority to identify a large number of sources. We consider $K = 34$ sources whose normalized DOAs are uniformly distributed over $[-0.45, 0.45]$, corresponding to DOAs in $[-2.58^\circ, 2.58^\circ]$. $N = 8$. The vertical lines mark real directions. A “Failed” text on the top-right corner in subfigures denotes that the spatial spectrum fails to identify all sources. The spatial spectra of the proposed method are more accurate compared to those of the methods in [31] and [34].

C. RMSE for 1-D DOA Estimation

For 1-D DOA estimation, the RMSE is computed as

$$
\text{RMSE} = \sqrt{\frac{1}{N_mK} \sum_{k=1}^{K} \sum_{i=1}^{N_m} (\hat{\theta}_{k,i} - \theta_k)^2}
$$

where $\hat{\theta}_{k,i}$ stands for the estimate of $\theta_k$ at the $i$th trial and $N_m$ is the number of Monte Carlo trials, respectively. In this article, $N_m = 200$.

First, we investigate the RMSE versus SNR. Ten sources are uniformly located between $-70^\circ$ and $70^\circ$. Fig. 8 depicts the RMSE of the three 1-D DOA estimation methods for $T = 3000$ snapshots when SNR ranges from $-10$ to $10$ dB with a step $2$ dB. It can be noted that for the same original array, the proposed method shows a smaller error in comparison to the methods in [31] and [34]. This is attributed to the higher number of uDOFs achieved by the proposed method. Generally, a higher number of uDOFs lead to a smaller error. In addition, for the same method, the smallest RMSEs can be obtained when using the ANA as the original array.

Then, we show the RMSE for the three methods and the three types of original arrays in Fig. 9 with SNR = 0 dB when the number of snapshots ranges from 300 to 5700 with a step 600. We set other parameters the same as those in Fig. 8. It can be noted that the result in Fig. 9 is similar to that in Fig. 8, i.e., the smallest error is obtained by the proposed method, followed by the method in [34], and then by the method in [31].
Fig. 7. Spatial spectra for $K = 34$ sources. The normalized DOAs are uniformly distributed over $[-0.45, 0.45]$, corresponding to DOAs in $[-64.16^\circ, 64.16^\circ]$. $N = 8$. The vertical lines mark real directions. The proposed method can identify all 34 sources for all different arrays while the method in [34] fails to identify all sources for the CPA. Besides, for the same original array, the proposed method shows sharp peaks compared to the method in [34].

Next, we investigate the RMSE as a function of the number of sources. Let the normalized DOAs of all sources be uniformly distributed over $[-0.45, 0.45]$. Note that the maximum number of detectable sources for the method in [31] with the CPA is 16. Therefore, we first plot the RMSE of the three methods with $\text{SNR} = 0 \text{ dB}$ and 3000 snapshots when the number of sources ranges from 1 to 16 in Fig. 10. First, it can be noted that the errors of the three methods increase as the number of sources increases. Second, the proposed method shows smaller error compared to the other two methods for the same array geometry. Basically, a larger number of uDOFs result in a smaller error.

In practice, the maximum number of detectable sources is affected by many factors, such as the number of snapshots, SNR, and the minimum space between two sources. To elaborately evaluate how many sources that the proposed method can identify, we increase the range of the number of sources and depict the RMSE versus the number of sources only for the proposed method with the three different array geometries in Fig. 11. It can be seen that when the number of sources is 52, the proposed method for the 5-DCPA obtains the largest RMSE among the three array geometries and it is lower than $0.4^\circ$. It shows that the proposed method with the three array geometries can successfully identify more than 50 sources. It is quite close to the theoretical limit considering that the theoretical maximum number of detectable sources for the 5-DCPA is 57.
D. Perturbation on the Speed

In practice, the sensor array may not move at a constant speed. Here, we present an experiment to investigate the performance of 1-D DOA estimation against the perturbation on the speed. Suppose that at time $t$, the moving speed of the sensor array is $v_t = v(1 + \Delta v)$, where $\Delta v$ is a Gaussian random variable and normally distributed with mean zero and variance $\sigma_v^2$. We illustrate the RMSE of 1-D angle estimates as a function of the standard deviation $\sigma_v$ for different array geometries in Fig. 12.

First, it can be clearly seen that for the same type of original array, the proposed method outperforms the two other methods when $\sigma_v \leq 0.3$. When the perturbation is small, the number of uDOFs achieved by these arrays dominates the performance of DOA estimation. Since the proposed method has the highest uDOF, it achieves the best performance for small perturbation case. Second, when the perturbation is large, i.e., $\sigma_v \geq 0.3$, the performance of the proposed method degrades significantly and is less robust compared with the other two methods. The large perturbation has more impact on the DOA estimation performance of the proposed method in comparison to that of the two other methods.

E. RMSE for 2-D DOA Estimation

In this section, we investigate the RMSEs of the proposed model for 2-D DOA estimation, which are computed as

$$
\text{RMSE}(\theta) = \sqrt{\frac{1}{N_s K} \sum_{i=1}^{N_s} \sum_{k=1}^{K} (\hat{\theta}_{k,i} - \theta_k)^2}
$$
$$
\text{RMSE}(\phi) = \sqrt{\frac{1}{N_s K} \sum_{i=1}^{N_s} \sum_{k=1}^{K} (\hat{\phi}_{k,i} - \phi_k)^2}
$$

where $\hat{\theta}_{k,i}$ and $\hat{\phi}_{k,i}$ denote the estimates of $\theta_k$ and $\phi_k$ at the ith trial.

We consider that ten sources with normalized azimuth and elevation uniformly distributed over $[-0.08, 0.135]$ and $[0.125, 0.35]$, impinge on all the above parallel arrays. We depict the RMSE versus SNR for different parallel arrays with 3000 snapshots in Fig. 13(a) and (b) when SNR ranges from $-10$ to $10$ dB with a step $2$ dB. It can be noted that the proposed method based on two-parallel 5-DAs obtains smaller errors compared with the methods in [34] and [45]. This is because the proposed method has the largest number of DOFs, i.e., it can resolve the largest number of sources among the three methods.

Next, we illustrate the RMSEs versus the number of snapshots when SNR = 0 dB in Fig. 14(a) and (b). We keep other
parameters the same as those in Fig. 13. It can be noted that the proposed method achieves the best performance as well among all methods due to increased DOFs.

VI. Conclusion

To increase the number of uDOFs for \( q \)-dilated arrays with dilation factor \( q \geq 4 \), in this article, we design a new moving array processing model in \( q \)-dilated arrays. The proposed model synthesizes multiple shifted arrays with displacements of half wavelength multiples for dilation factor \( q \geq 4 \), leading to a possibility to increase the numbers of DOFs and uDOFs. First, we show that the numbers of the DOFs and uDOFs achieved by \( q \)-dilated arrays are the functions of the dilation factor \( q \), the number of shifted arrays \( R \), and the geometry of their original arrays. We prove that the numbers of the DOFs and uDOFs achieved \( q \)-dilated arrays are \( q \) times those of their original arrays for arbitrary positive integer \( q \). Thus, the number of detectable sources is increased by a factor of \( q \) as well.

Second, we apply the proposed model to two-parallel dilated arrays for 2-D DOA estimation, increasing the maximum number of sources by a factor of \( q \). In addition, we provide a way to acquire a fully filled difference coarray for \( q \)-dilated arrays, making it convenient to apply ULA-based DOA estimation approaches.

Numerical experiments are conducted to compare the proposed model with state-of-the-art methods in terms of spatial spectra and RMSE for three types of array geometries, i.e., CPA, NA, and ANA. It is shown that the proposed model obtains more accurate spatial spectra and achieves smaller RMSEs for 1-D DOA estimation compared to the methods in [31] and [34] due to increased uDOFs. Meanwhile, for 2-D DOA estimation, the proposed model has better performance as well in comparison to the methods in [34] and [45].

In the proposed moving array model, the physical array moves uniformly. The number of uDOFs may be further increased by moving the array nonuniformly. This will be investigated in future work. Also note that in practice, the array may not move along the linear array accurately. The movement deviation issue is another interest and meaningful topic to be solved. In real-world scenarios, the multipath issue often happens. Thus, it is meaningful to investigate the multipath issue for the sparse arrays and moving sparse arrays in future work.
APPENDIX A
PROOF OF LEMMA 1

Plugging (11) into $C(N_q^i, N_q^j)$, we have

$$C(N_q^i, N_q^j) = C(N_q^i + i, N_q^j + r).$$

(25)

Following the properties of the difference set, we have

$$C(N_q^i, N_q^j) = C(N_q^i + r + i - r, N_q^0 + r)$$

$$= D(N_q^i + r) + i - r$$

$$= D(N_q^i) + i - r.$$  

(26)

Lemma 1 has been proved.

APPENDIX B
PROOF OF THEOREM 1

According to (2) and (12), we have

$$D(S_q^R) = D(N_q^0 \cup N_q^1 \cup \ldots \cup N_q^{R_q})$$

$$= \bigcup_{r = 0}^{R_q} D(N_q^r) \bigcup \bigcup_{i = 0}^{R_q} C(N_q^i, N_q^j).$$

(27)

According to Lemma 1, we arrive at

$$D(S_q^R) = D(N_q^0) \bigcup \bigcup_{i = 0}^{R_q} D(N_q^i) + i - r$$

$$= \bigcup_{r = -R_q}^{R_q} D(N_q^r) + r.$$  

(28)

Note that $D(N_q^0) = D(qL) = qD(L)$. Therefore, we have

$$D(S_q^R) = \bigcup_{r = -R_q}^{R_q} [qD(L) + r].$$

(29)

Thus, Theorem 1 is proved.

APPENDIX C
PROOF OF THEOREM 2

According to Theorem 1, we know that each element in $D(L)$ produces $2R + 1$ corresponding elements in $D(S_q^R)$. For $|D(L)|$ elements in $D(L)$, $(2R + 1)|D(L)|$ elements can be obtained. However, these elements may overlap.

Assume that $D(L)$ has two integer elements $a$ and $b$, where $a < b$. Then, from Theorem 1, $D(S_q^R)$ has the following elements: $qa, qa \pm 1, \ldots, qa \pm R, qb, qb \pm 1, \ldots, qb \pm R$. Let $Z_a = \{qa, qa \pm 1, \ldots, qa \pm R\}$ and $Z_b = \{qb, qb \pm 1, \ldots, qb \pm R\}$.

Consider the difference between the maximum element in $Z_a$ and the minimum one in $Z_b$, i.e., $qa + R$ and $qb - R$.

A. Case 1

If $R < (q - 1)/2$, then

$$(qa + R) - (qb - R) < q(a - b) + q - 1 < -1$$

(30)

which implies that any two elements in the two sets $Z_a$ and $Z_b$ do not overlap and at least one hole appears between $Z_a$ and $Z_b$. Hence, we conclude that none of the $(2R + 1)|D(L)|$ elements in $D(S_q^R)$ overlap with each other, that is to say

$$|D(S_q^R)| = (2R + 1)|D(L)|.$$

(31)

In the corresponding difference coarray, the longest ULA only has $2R + 1$ sensors, that is

$$|U(S_q^R)| = 2R + 1.$$  

(32)

B. Case 2

If $q$ is even and $R = q/2$, then

$$(qa + R) - (qb - R) = q(a - b + 1) \leq 0.$$  

(33)

The equation holds if and only if $a$ and $b$ are consecutive integers. It indicates that two of the $(2R + 1)|D(L)|$ elements overlap with each other for each two contiguous integers in $D(L)$. In $D(L)$, there are at most $|D(L)| - 1$ pairs of elements overlapping with each other. Hence,

$$|D(S_q^R)| \geq (2R + 1)|D(L)| - (|D(L)| - 1)$$

$$= q|D(L)| + 1.$$  

(34)

For $U(S_q^R)$, since there are $|U(L)| - 1$ pairs of consecutive elements in $U(L)$, we have

$$U(S_q^R) = q|U(L)| + 1.$$  

(35)

C. Case 3

For odd $q$, if $R = (q - 1)/2$, then

$$(qa + R) - (qb - R) = q(a - b + 1) - 1 \leq -1.$$  

(36)

The equation holds if and only if $a$ and $b$ are consecutive integers. It implies that none of the $(2R + 1)|D(L)|$ elements overlap, i.e.,

$$|D(S_q^R)| = (2R + 1)|D(L)| = q|D(L)|.$$  

(37)

Meanwhile, when $a$ and $b$ are consecutive integers, the elements composed of the corresponding sets $Z_a$ and $Z_b$ exactly constitute a ULA. Thus,

$$|U(S_q^R)| = q|U(L)|.$$  

(38)

D. Case 4

Consider $a$ and $b$ are two consecutive integers, i.e., $a - b = -1$. If $R > q/2$, then

$$(qa + R) - (qb - R) = 2R - q > 0$$

(39)

which indicates that there is at least one element in $Z_a$ overlapping with an element in $Z_b$, if $a$ and $b$ are consecutive.

Note that $|D(S_q^R)|$ and $|U(S_q^R)|$ are two monotonically increasing functions of $R$. When $R$ increases by one, at least two new elements are added to $D(S_q^R)$ and $U(S_q^R)$. For $|D(S_q^R)|$, if $q$ is odd

$$|D(S_q^R)| \geq |D(S_q^{(q-1)/2})| + 2(R - (q - 1)/2)$$

$$= q|D(L)| + 2R - q + 1.$$  

(40)

If $q$ is even, we obtain

$$|D(S_q^R)| \geq |D(S_q^{q/2})| + 2(R - q/2)$$

$$\geq q|D(L)| + 2R - q + 1.$$  

(41)
For all \( q \), we, therefore, have
\[
\left| D \left( S_{q}^{R} \right) \right| \geq q |D(L)| + 2R - q + 1. \tag{42}
\]
For \( |U(S_{q}^{R})| \), if \( q \) is odd, then
\[
\left| U \left( S_{q}^{R} \right) \right| \geq \left| U \left( S_{q}^{q-1/2} \right) \right| + 2(R - (q - 1)/2) = q |U(L)| + 2R - q + 1. \tag{43}
\]
If \( q \) is even, then
\[
\left| U \left( S_{q}^{R} \right) \right| \geq q |U(L)| + 2R - q + 1. \tag{44}
\]
The same result is obtained for even and odd \( q \). Hence, for all \( q \), we have
\[
\left| U \left( S_{q}^{R} \right) \right| \geq q |U(L)| + 2R - q + 1. \tag{45}
\]
In summary, Theorem 2 has been proved.

**APPENDIX D**

**Proof of Corollary 2**

If \( q \) is odd, then \( R = \lfloor q/2 \rfloor = (q - 1)/2 \). According to Table 1, \( |D(S_{q}^{R})| = q |D(L)| \).

If \( q \) is even, then \( R = |q/2| = q/2 \). As shown in the proof of case 2 in Appendix C, for each two contiguous integers in \( D(L) \), two of the \((2R + 1) |D(L)| \) elements overlap with each other. Let \( I_{i} \) be the number of elements of the \( i \)th island. Note that there are \( I_{i} - 1 \) pairs of neighboring sensors in the \( i \)th island. Thus, we have
\[
\left| D \left( S_{q}^{R} \right) \right| = (2R + 1) |D(L)| - \sum_{i=1}^{I_{i}} (I_{i} - 1). \tag{46}
\]
Note that \( \sum_{i=1}^{I_{i}} I_{i} = |D(L)| \). Therefore,
\[
\left| D \left( S_{q}^{R} \right) \right| = q |D(L)| + I_{D}. \tag{47}
\]
Hence, Corollary 2 has been proved.

**APPENDIX E**

**Proof of Theorem 3**

First, we prove that when \( R = |q/2| \) and \( D(L) \) is hole-free, the corresponding synthetic array \( S_{q}^{R} \) can acquire a hole-free difference coarray with \( |D(S_{q}^{R})| = q |D(L)| + 2R - q + 1 \) elements.

Because \( D(L) \) is hole-free, we arrive at
\[
|U(L) = D(L). \tag{48}
\]
For odd \( q \), if \( R = \lfloor q/2 \rfloor \), then \( R = (q - 1)/2 \). According to Table 1
\[
\left| D \left( S_{q}^{R} \right) \right| = \left| U \left( S_{q}^{R} \right) \right| = q |D(L)| \tag{49}
\]
which means that \( D(S_{q}^{R}) \) is a hole-free ULA when \( R = (q - 1)/2 \).

For even \( q \), if \( R = |q/2| \), then \( R = q/2 \). From Table 1
\[
\left| U \left( S_{q}^{R} \right) \right| = q |U(L)| + 1 = q |D(L)| + 1. \tag{50}
\]
Note that the number of islands in \( D(L) \) is 1 since \( D(L) \) is a hole-free ULA. According to Corollary 2, we obtain
\[
\left| D \left( S_{q}^{R} \right) \right| = q |D(L)| + 1. \tag{51}
\]
By combining (50) and (51), we have
\[
\left| D \left( S_{q}^{R} \right) \right| = \left| U \left( S_{q}^{R} \right) \right| = q |D(L)| + 1 \tag{52}
\]
which indicates that for \( R = |q/2| \), \( D(S_{q}^{R}) \) is a hole-free ULA as well.

When \( R = |q/2| \), (49) and (52) can be rewritten as
\[
\left| D \left( S_{q}^{R} \right) \right| = q |D(L)| + 2R - q + 1. \tag{53}
\]
Thus, it is proved that when \( R = |q/2| \) and \( D(L) \) is a ULA, the corresponding synthetic array \( S_{q}^{R} \) has a hole-free difference coarray with \( |D(S_{q}^{R})| = q |D(L)| + 2R - q + 1 \) elements.

Then, consider that \( R \geq |q/2| \). Note that if \( D(S_{q}^{R}) \) is a hole-free ULA, \( D(S_{q}^{R+1}) \) is also a ULA with the number of sensors
\[
\left| D \left( S_{q}^{R+1} \right) \right| = \left| D \left( S_{q}^{R} \right) \right| + 2. \tag{54}
\]
Hence, if \( R \geq |q/2| \), \( D(S_{q}^{R}) \) is a ULA. By combining (53) and (54), the number of sensors in \( D(S_{q}^{R}) \) for \( R \geq |q/2| \) is
\[
\left| D \left( S_{q}^{R} \right) \right| = q |D(L)| + 2R - q + 1. \tag{55}
\]
Thus, Theorem 3 is proved.

**REFERENCES**


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