Signal and Systems I

Lecture 7

Last Lecture

- LTI System (Stability test & Causality test)
- Convolution Properties
- LTI System interconnections

Today

- Fourier Series

Fourier Series

$x(t)$: periodic signal with fundamental period T_0
(Fundamental freq. $\omega_0 = \frac{2\pi}{T_0}$) can be written as sum of periodic spirals:

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 nt} \qquad \text{(Fourier series)} \Longleftarrow \textit{Synthesis}$$

$$D_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-j\omega_0 nt}dt \qquad \text{(Inverise FS, Finds } D_n \text{ from } x(t)) \Longleftarrow \textit{Analysis}$$

D_n in the above equation is *countable*.

D_0 is the signal bias (Direct Current (DC)) of $x(t)$.

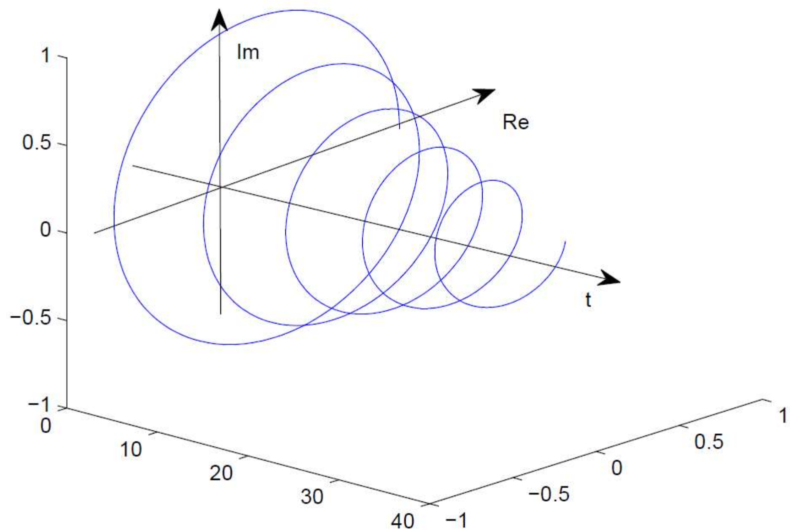
$$D_n = |D_n|e^{j\angle D_n}$$

$|D_n|$, absolute value of D_n, shows the amplitude of the periodic spiral $e^{j\omega_0 nt}$
$\angle D_n$, angle of D_n, is the amount of rotation of the spiral.

Fourier Series

Fourier Series are built with spirals. But only with periodic spirals to synthesis periodic signals.

$x(t) = e^{st}, \quad s = -2 + j\pi, \alpha = -2 \neq 0$, since α is non-zero, this function is **not periodic! and therefore has no Fourier series.**



Fourier Series

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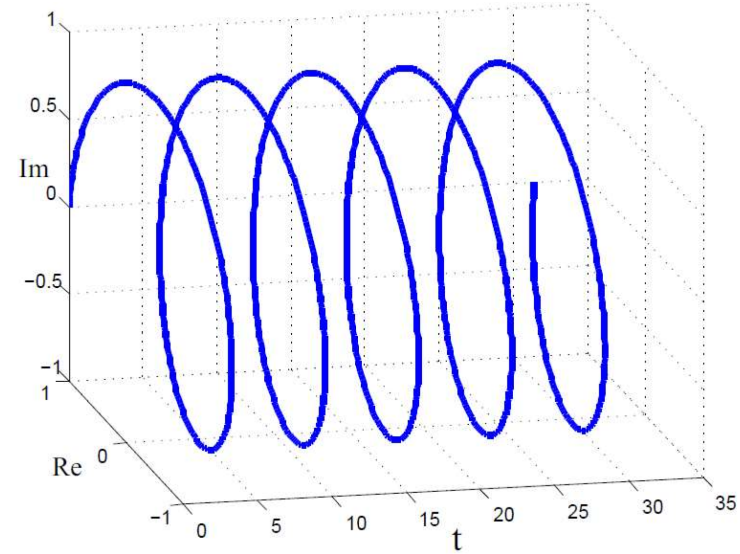
$$D_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-j\omega_0 nt}dt \qquad \text{(Finds } D_n \text{ from } x(t)) \Longleftarrow \textit{Analysis}$$

$$x_1(t) = e^{j2\pi t}$$

$$x_1(t) = x_1(t+T_0)$$
$$e^{j2\pi t} = e^{j2\pi(t+1)} = e^{j2\pi t}\underbrace{e^{j2\pi}}_{1} \Rightarrow T_0 = 1$$



This signal is periodic.
In this example $x(t) = \underbrace{1e^{j\omega_0 t}}_{\text{Fourier series}} = D_1 \underbrace{e^{j\omega_0 1t}}_{n=1}, \ \ D_1 = 1$

$$x_2(t) = je^{j2\pi t}$$

$$x_2(t) = x_2(t+T_0)$$
$$je^{j2\pi t} = je^{j2\pi(t+1)} = je^{j2\pi t}\underbrace{e^{j2\pi}}_{1} \Rightarrow T_0 = 1$$

This signal is periodic.
In this example $x(t) = \underbrace{je^{j\omega_0 t}}_{\text{Fourier series}} = D_1 \underbrace{e^{j\omega_0 1t}}_{n=1}, \ \ D_1 = j \qquad |D_1| = 1, \ \angle(D_1) = \frac{\pi}{2}$

Fourier Series

Example: Can we have Fourier series for $x(t) = \frac{1}{2}e^{-j\frac{2\pi}{3}t}$?
Answer: first check if this function is periodic:

$$x(t) = x(t+T_0)$$
$$\frac{1}{2}e^{-j\frac{2\pi}{3}t} = \frac{1}{2}e^{-j\frac{2\pi}{3}(t+T_0)}$$
$$e^{-j\frac{2\pi}{3}t} = e^{-j\frac{2\pi}{3}t}.e^{-j\frac{2\pi}{3}T_0}$$

If we choose $e^{-j\frac{2\pi}{3}T_0} = 1 \rightarrow T_0 = 3$ (signals in form of $e^{j\omega_0 t}$ are periodic with fundamental period ω_0).

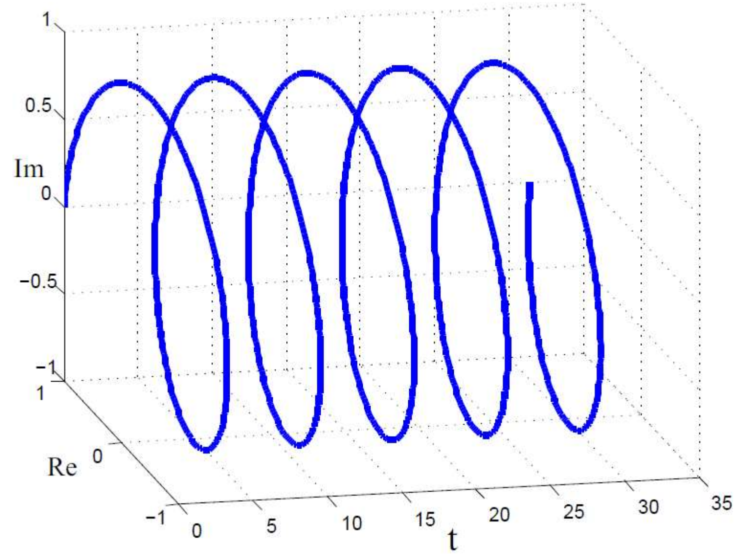
$$x(t) = \sum D_n e^{j\omega_0 nt} = \sum D_n e^{j\frac{2\pi}{3}nt}$$
$$\frac{1}{2}e^{-j\frac{2\pi}{3}t} = ... + D_{-2}e^{-j\frac{2\pi}{3}2t} + D_{-1}e^{-j\frac{2\pi}{3}1t} + D_0 + D_1 e^{j\frac{2\pi}{3}1t} + D_2 e^{j\frac{2\pi}{3}2t} + ... \quad \begin{cases} D_{-2} = 0 \\ D_{-1} = \frac{1}{2} \\ D_0 = 0 \\ D_1 = 0 \end{cases} \Rightarrow \quad \text{only} \quad D_{-1} = \frac{1}{2}$$

Fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 nt} \qquad \text{(Fourier series)} \Longleftarrow \textit{Synthesis}$$

$$D_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-j\omega_0 nt}dt \qquad \text{(Finds } D_n \text{ from } x(t)) \Longleftarrow \textit{Analysis}$$

$$D_n e^{jn\omega_0 t}$$



Componencts of Fourier Series are periodic spirals in form of $e^{jn\omega_0 t}$ which is a periodic spiral with frequency $n\omega_0$.

Each spiral is then rotated by angle of D_n and amplified by $|D_n|$

$x_1(t) = je^{-j5t}$

$\omega_0 = -5, \ \ |D_{-1}| = 1, \ \angle(D_{-1}) = \frac{\pi}{2}$

$x_2(t) = 2e^{j\frac{\pi}{3}}e^{j2\pi t}$

$\omega_0 = 2\pi, \ \ |D_1| = 2, \ \angle(D_1) = \frac{\pi}{3}$

$x_3(t) = -2e^{j\frac{5}{6}\pi t}$

$\omega_0 = \frac{5}{6}\pi, \ \ |D_1| = 2, \ \angle(D_1) = \pi$

Fourier Series

Example:

What is the Fourier series for $x(t) = 2sin(2t)$?
<u>Answer:</u> First find T_0 and ω_0

$$
\begin{aligned}
x(t) &= x(t+T_0) \\
2sin(2t) &= 2sin(2t+2T_0) \\
2T_0 &= 2\pi \\
T_0 &= \pi \\
\omega_0 &= \frac{2\pi}{T_0} = 2
\end{aligned}
$$

$$
\begin{cases} D_{-1} = \frac{-1}{j} = j \\ D_1 = \frac{1}{j} = -j \end{cases} \Rightarrow \quad \text{Other coefficients are all zero}
$$

$$
\begin{aligned}
x(t) &= \sum D_n e^{j\omega_0 nt} = \sum D_n e^{j2nt} \\
&= ... + D_{-2}e^{-j2.2t} + D_{-1}e^{-j2.1t} + D_0 + D_1 e^{j2.1t} + D_2 e^{j2.2t} + ...
\end{aligned}
$$

$$
2sin(2t) = 2\left(\frac{e^{j2t} - e^{-2jt}}{2j}\right) = \frac{-1}{j}e^{-j2t} + \frac{1}{j}e^{j2t}
$$

Fourier Series

Euler Formulas:

$$\cos(\alpha t + \beta) = \frac{e^{j(\alpha t+\beta)} + e^{-j(\alpha t+\beta)}}{2}$$

$$\sin(\alpha t + \beta) = \frac{e^{j(\alpha t+\beta)} - e^{-j(\alpha t+\beta)}}{2j}$$

Associated FS coefficients: $\omega_0 = \alpha$

$$\cos(\alpha t + \beta) = \underbrace{\frac{1}{2}e^{\beta}}_{D_1} e^{j(\alpha t)} + \underbrace{\frac{1}{2}e^{j\beta}}_{D_{-1}} e^{j(-\alpha t)}$$

$$\sin(\alpha t + \beta) = \underbrace{\frac{1}{j2}e^{\beta}}_{D_1} e^{j(\alpha t)} + \underbrace{\frac{-1}{2j}e^{j\beta}}_{D_{-1}} e^{j(-\alpha t)}$$

$\rightarrow$ One Spiral = One Coefficient

$\rightarrow$ One Sine/Cosine = Two Coefficients (one positive and one negative: $D_1 \& D_{-1}$

Fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 n t} \qquad \text{(Fourier series)} \Longleftarrow \textit{Synthesis}$$

$$D_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-j\omega_0 n t} dt \qquad \text{(Finds } D_n \text{ from } x(t)) \Longleftarrow \textit{Analysis}$$

Fourier transform of $e^{j\alpha t}$:

e^{j2t}, $\alpha = 2 = \omega_0$
$e^{j\omega_0 t} = D_1 e^{j\omega_0 1 t} \Rightarrow n = 1, D_1 = 1$, all other D_ns are zero.

Fourier Transform of $cos(\alpha t + \beta)$:

$$\begin{aligned} cos(2t + \frac{\pi}{2}) &= \frac{1}{2}e^{j(2t+\frac{\pi}{2})} + \frac{1}{2}e^{-j(2t+\frac{\pi}{2})} \qquad \omega_0 = 2 \\ &= \frac{1}{2}e^{j\frac{\pi}{2}}e^{j2t} + \frac{1}{2}e^{-j\frac{\pi}{2}}e^{-j2t} \\ &= \underbrace{\frac{1}{2}e^{j\beta}}_{D_1} e^{j\omega_0 t} + \underbrace{\frac{1}{2}e^{-j\beta}}_{D_{-1}} e^{-j\omega_0 t} \end{aligned}$$

Fourier Series

How to plot D_is:

$$D = |D|e^{j\angle(D)}$$

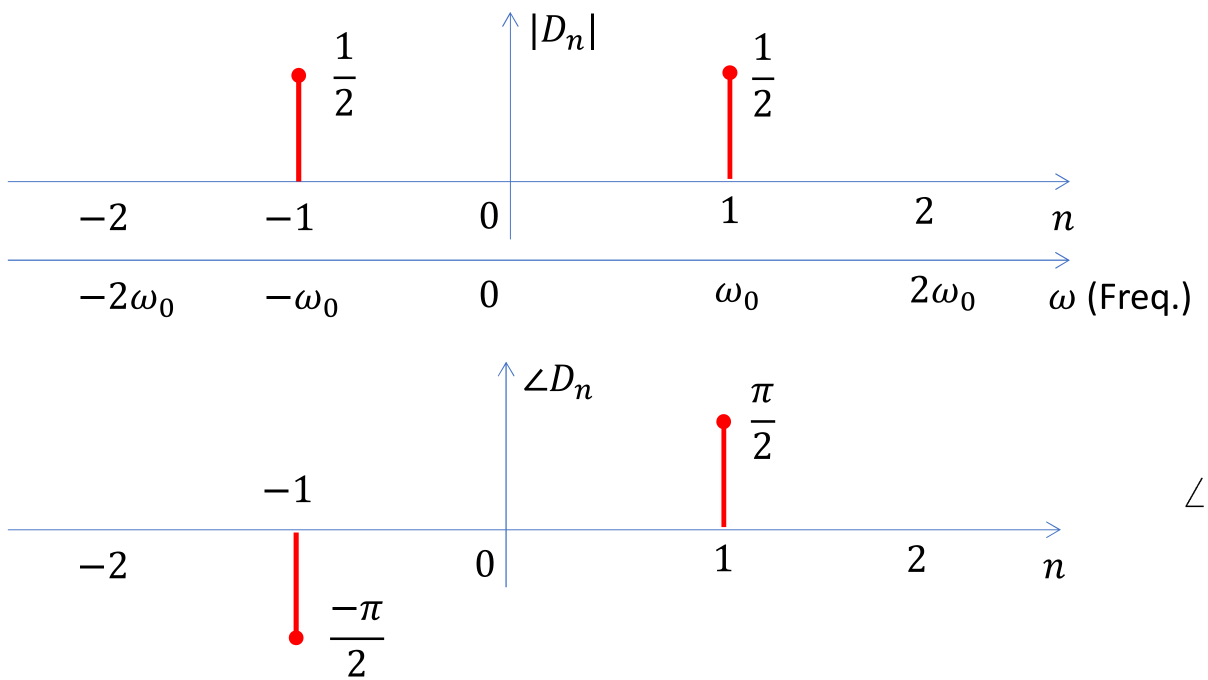
$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 nt} \qquad \text{(Fourier series)} \Longleftarrow \textit{Synthesis}$$

$$D_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-j\omega_0 nt}dt \qquad \text{(Finds } D_n \text{ from } x(t)) \Longleftarrow \textit{Analysis}$$

$$|\frac{1}{2}e^{j\frac{\pi}{2}}| = \frac{1}{2} \qquad \angle\frac{1}{2}e^{j\frac{\pi}{2}} = \frac{\pi}{2}$$

$$|\frac{1}{2}e^{-j\frac{\pi}{2}}| = \frac{1}{2} \qquad \angle\frac{1}{2}e^{-j\frac{\pi}{2}} = -\frac{\pi}{2}$$

$$\cos(2t + \frac{\pi}{2}) = \underbrace{\frac{1}{2}e^{j\frac{\pi}{2}}}_{D_1} e^{j\omega_0 t} + \underbrace{\frac{1}{2}e^{-j\frac{\pi}{2}}}_{D_{-1}} e^{-j\omega_0 t}$$



$\angle D_n$ is shown in $radian/second$

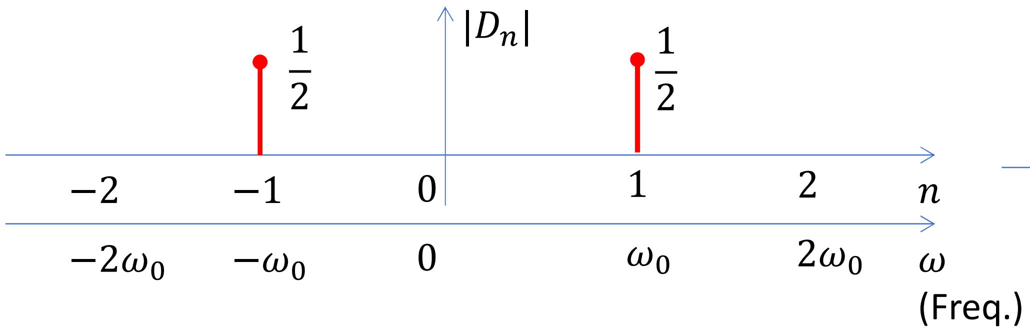
Fourier Series

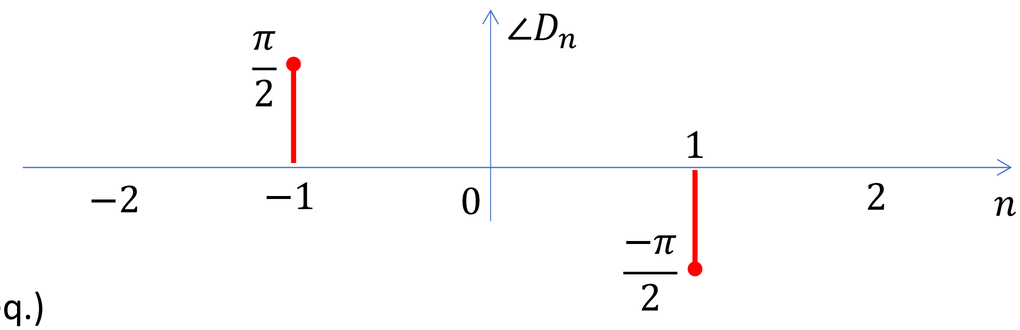
Fourier series of $\sin(\alpha t)$:

$$sin(\frac{\pi}{3}t) = \underbrace{\frac{1}{2j}}_{D_1} e^{j(\frac{\pi}{3}t)} - \underbrace{\frac{1}{2j}}_{D_{-1}} e^{-j(\frac{\pi}{3}t)} \qquad \omega_0 = \frac{\pi}{3}$$

$$|D_1| = |\frac{1}{2j}| = \frac{1}{2} \qquad \angle D_1 = \angle\frac{1}{2j} = \angle\frac{1}{j} = \angle\frac{1}{j} \times \frac{j}{j} = \angle\frac{j}{j \times j} = \angle -j = \frac{-\pi}{2}$$

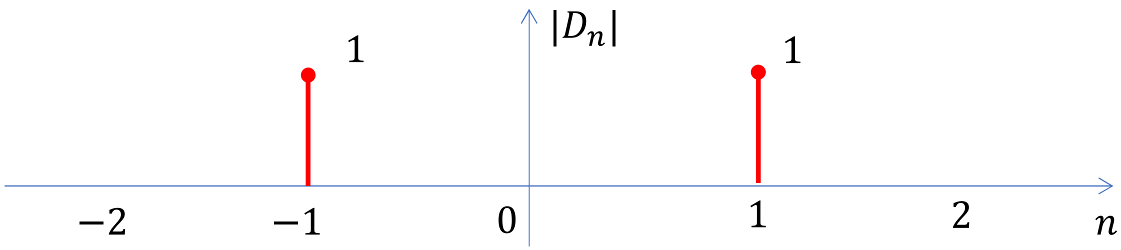
$$|D_{-1}| = |\frac{-1}{2j}| = \frac{1}{2} \qquad \angle D_{-1} = \angle\frac{-1}{2j} = \angle\frac{-1}{j} \times \frac{j}{j} = \angle\frac{-j}{j \times j} = \angle j = \frac{\pi}{2}$$





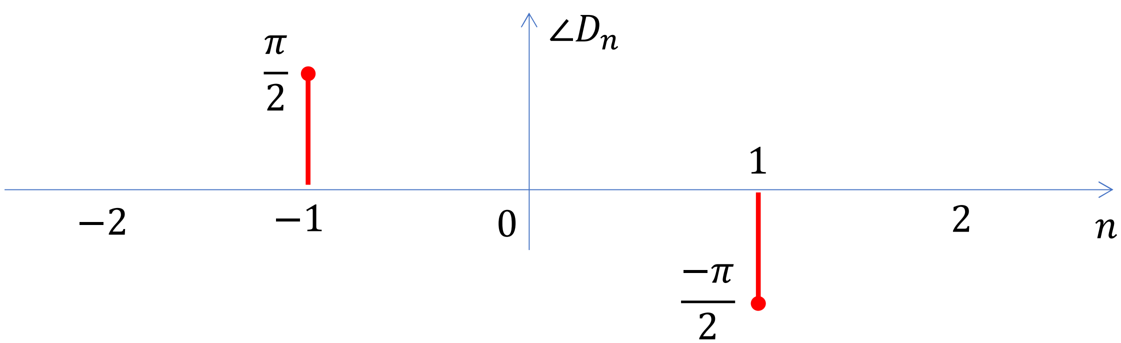
Fourier Series

Example: Fourier series of $x(t)$ is as follows. Find $x(t)$. $(\omega_0 = 2)$



$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 n t} \qquad \text{(Fourier series)} \Longleftarrow \textit{Synthesis}$$

$$D_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-j\omega_0 n t} dt \qquad \text{(Finds } D_n \text{ from } x(t)) \Longleftarrow \textit{Analysis}$$



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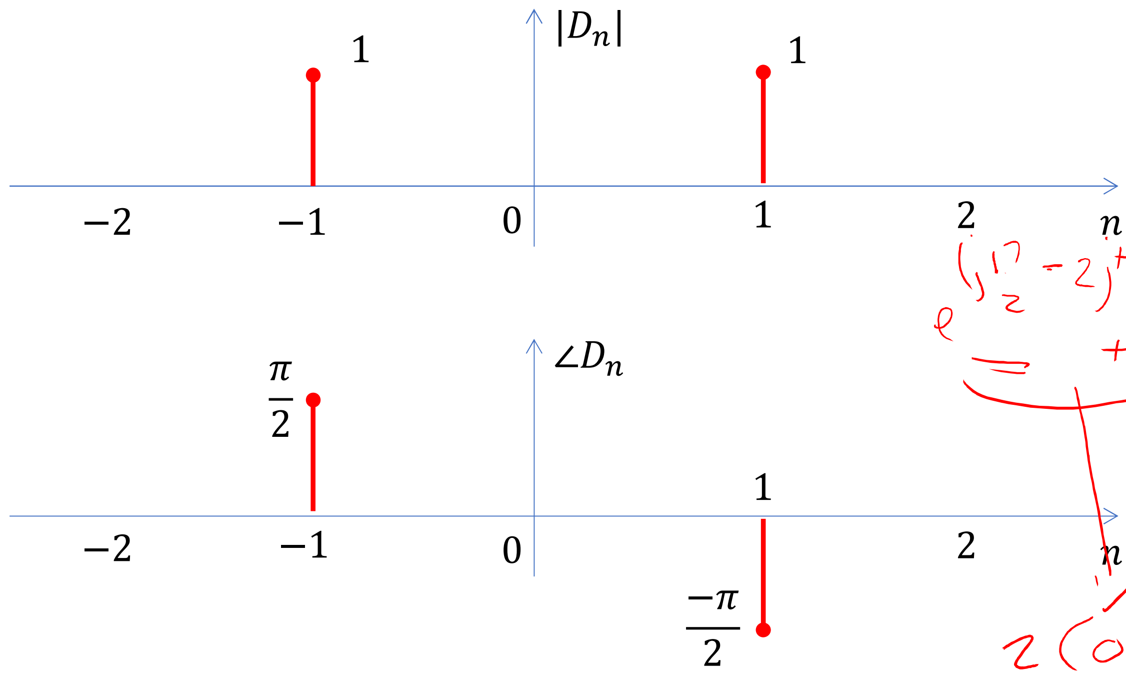


$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 n t}$$

$$= D_{-1}e^{-j2t} + D_1 e^{j2t}$$

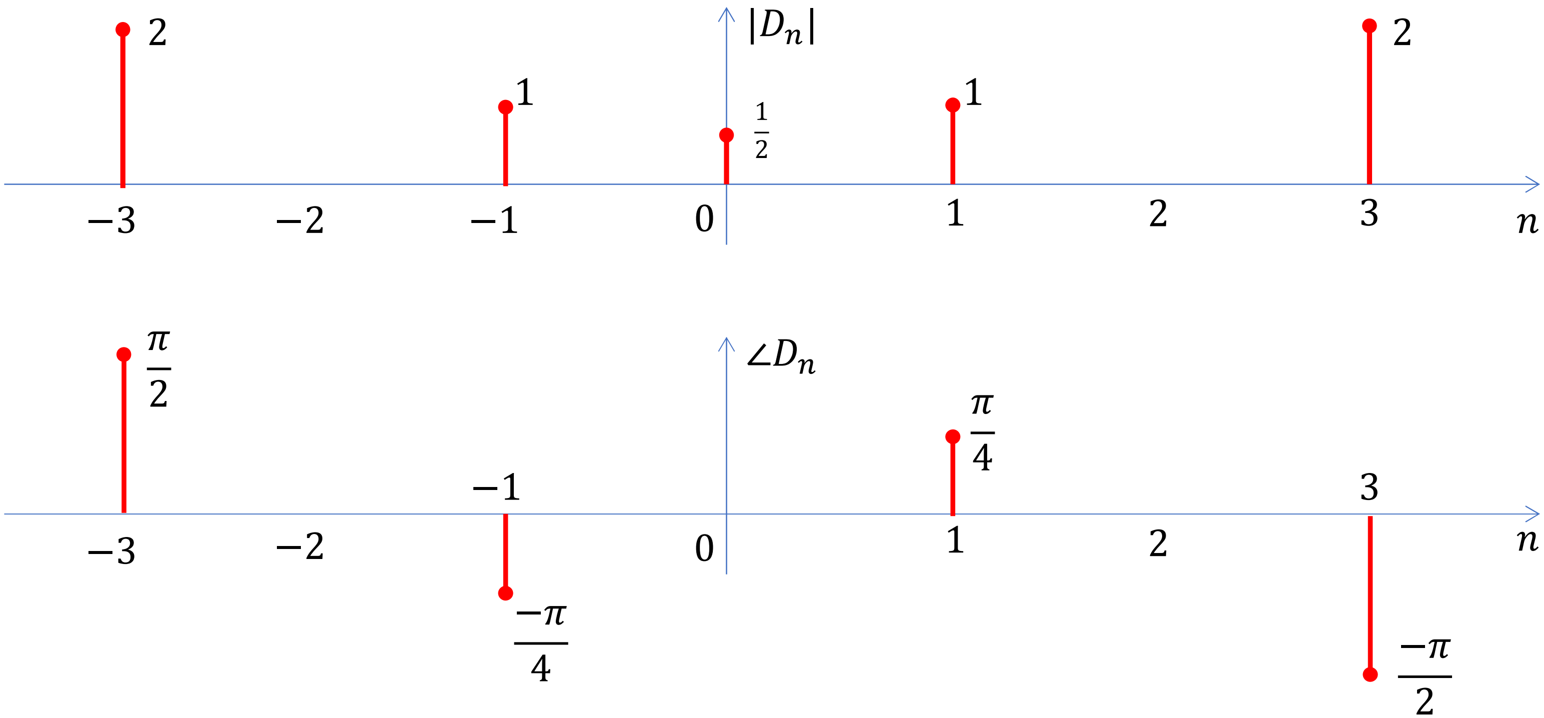
$$= 1.e^{j\frac{\pi}{2}}e^{-j2t} + 1.e^{-j\frac{\pi}{2}}e^{j2t}$$

$$= je^{-j2t} - je^{j2t}$$

$$= \frac{e^{j2t} - e^{-j2t}}{j} = 2\sin(2t)$$

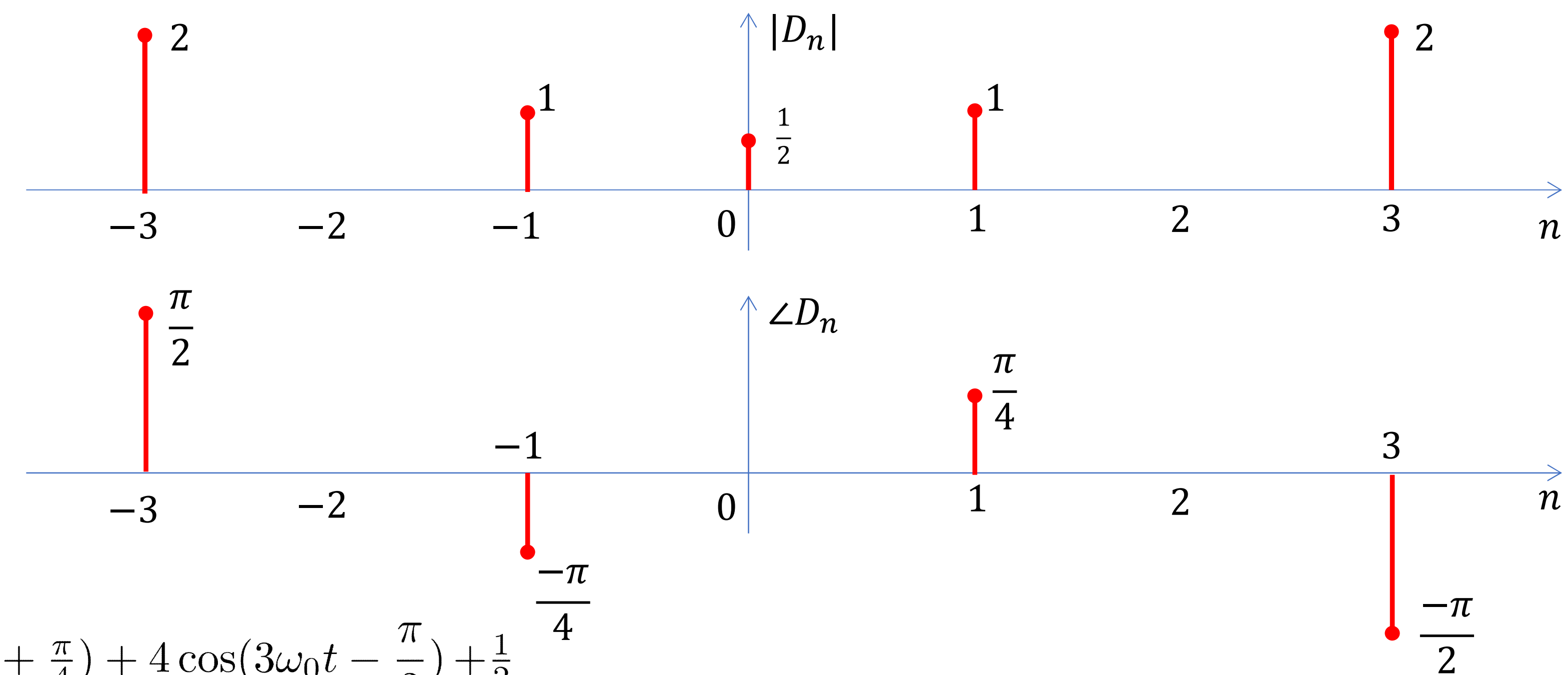
Fourier Series

Example: Find $x(t)$ from its Fourier series coefficients.



Fourier Series

Example: Find $x(t)$ from its Fourier series coefficients.



Answer:

$$2\cos(\omega_0 t + \tfrac{\pi}{4}) + \underbrace{4\cos(3\omega_0 t - \frac{\pi}{2})}_{4\sin(3\omega_0 t)} + \tfrac{1}{2}$$

Note: Unlike the previous example ω_0 is not provided in this example, So $x(t)$ is provided as a function of ω_0.

Fourier Series

Example: Find Fourier series of the following signal $x(t) = sin(\frac{2\pi}{3}t) + cos(\frac{\pi}{5}t)$.
(Note: Since the function has only sin and cos function, Euler formula is used.)

First check if the signal is periodic!

Fourier Series

Example: Find Fourier series of the following signal $x(t) = sin(\frac{2\pi}{3}t) + cos(\frac{\pi}{5}t)$.
(Note: Since the function has only *sin* and *cos* function, Euler formula is used.)

Solution: The first step is to find the period of this signal.
Fundamental Freq of $\sin(\frac{2\pi}{3}t)$: $\omega_0 = \frac{2\pi}{3}$
Fundamental Freq of $\cos(\frac{\pi}{5}t)$: $\omega_0 = \frac{\pi}{5}$

Adding two periodic signal does not always result in a periodic signal. So we need to first check if $x(t)$ is periodic or not.

$$x(t + T_0) = x(t)$$

$$sin\left(\frac{2\pi}{3}(t + T_0)\right) + cos\left(\frac{\pi}{5}(t + T_0)\right) = sin(\frac{2\pi}{3}t) + cos(\frac{\pi}{5}t)$$

$$\begin{cases} \frac{2\pi}{3}(t + T_0) = \frac{2\pi}{3}t + 2\pi k_1 \\ \frac{\pi}{5}(t + T_0) = \frac{\pi}{5}t + 2\pi k_2 \end{cases} \Rightarrow \begin{cases} \frac{2\pi}{3}T_0 = 2\pi k_1 \\ \frac{\pi}{5}T_0 = 2\pi k_2 \end{cases} \Rightarrow \begin{cases} \frac{2\pi}{3}T_0 = 2\pi k_1 \\ \frac{\pi}{5}T_0 = 2\pi k_2 \end{cases} \Rightarrow \begin{cases} T_0 = 3k_1 \\ T_0 = 10k_2 \end{cases}$$

T_0: Smallest multiple of 3 & 10. (Lowest common multiple (LCM) of 3 & 10)
$T_0 = 3 \times 10 = 30 \Rightarrow \omega_0 = \frac{2\pi}{30}$

Fourier Series

Now write the Euler formula for $x(t)$:

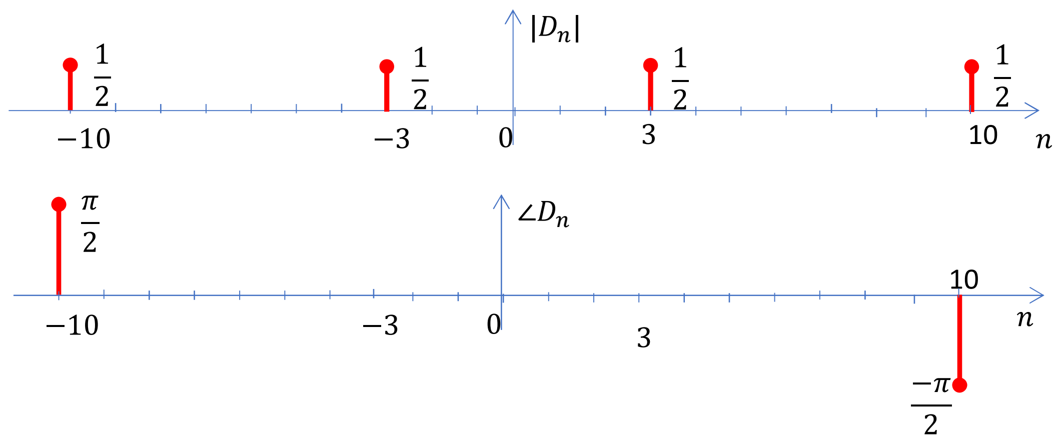
$$x(t) = \sin(\frac{2\pi}{3}t) + \cos(\frac{\pi}{5}t)$$

$$= \frac{e^{j\frac{2\pi}{3}t} - e^{-j\frac{2\pi}{3}t}}{2j} + \frac{e^{j\frac{2\pi}{10}t} + e^{-j\frac{2\pi}{10}t}}{2}$$

Powers $\quad \frac{2\pi}{3} \quad \frac{-2\pi}{3} \quad \frac{2\pi}{10} \quad \frac{-2\pi}{10}$

$\omega_0 = \frac{2\pi}{30} \quad \omega_0 \times 10 \quad \omega_0 \times (-10) \quad \omega_0 \times 3 \quad \omega_0 \times (-3)$

$$x(t) = \underbrace{\frac{1}{2j}}_{D_{10}} e^{j\omega_0 10t} \underbrace{-\frac{1}{2j}}_{D_{-10}} e^{-j\omega_0 10t} \underbrace{+\frac{1}{2}}_{D_3} e^{j\omega_0 3t} \underbrace{+\frac{1}{2}}_{D_{-3}} e^{-j\omega_0 3t}$$



Fourier Series

Example: Find Fourier series of the following signal and plot D_ns.

$$x(t) = -2 + \sin(\frac{2\pi t}{3}) + 2\cos(\frac{\pi t}{9})$$

Fourier Series

Example: Find Fourier series of the following signal and plot D_ns.

$$x(t) = -2 + \sin(\frac{2\pi t}{3}) + 2\cos(\frac{\pi t}{9})$$

Solution:

First we find the period of the signal

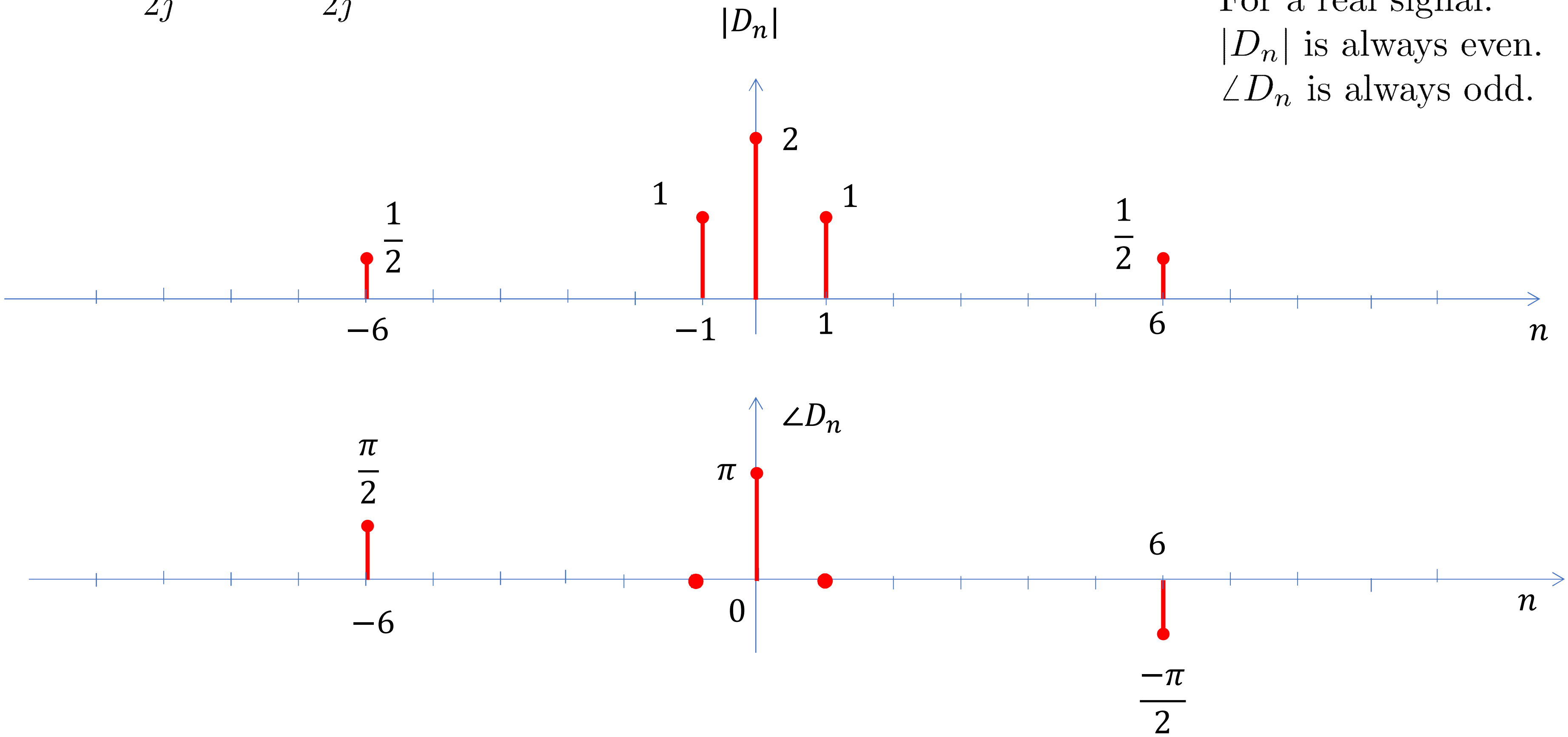
$$\begin{cases} \frac{2\pi}{3}T_0 = k_1.2\pi \\ \frac{\pi}{9}T_0 = k_2.2\pi \end{cases} \Rightarrow \begin{cases} T_0 = 3k_1 \\ T_0 = 18k_2 \end{cases} \Rightarrow \text{LCM of } (3 \text{ \& } 18) \Rightarrow T_0 = 18 \Rightarrow \omega_0 = \frac{2\pi}{18} = \frac{\pi}{9}$$

Now, write the Euler expansion of the given signal:

$$\begin{aligned} x(t) &= -2 + \frac{e^{j\frac{2\pi}{3}t} - e^{-j\frac{2\pi}{3}t}}{2j} + 2\frac{e^{j\frac{\pi}{9}t} + e^{-j\frac{\pi}{9}t}}{2} \\ &= -2 + \frac{1}{2j}e^{j\frac{2\pi}{3}t} - \frac{1}{2j}e^{-j\frac{2\pi}{3}t} + e^{j\frac{\pi}{9}t} - e^{-j\frac{\pi}{9}t} \\ &= -2 + \frac{1}{2j}e^{j\omega_0.6t} - \frac{1}{2j}e^{j\omega_0.(-6)t} + e^{j\omega_0 t} + e^{j(-\omega_0)t} \end{aligned}$$

Fourier Series

$$x(t) = -2 + \frac{1}{2j}e^{j\omega_0.6t} - \frac{1}{2j}e^{j\omega_0.(-6)t} + e^{j\omega_0 t} + e^{j(-\omega_0)t}$$



Important note:

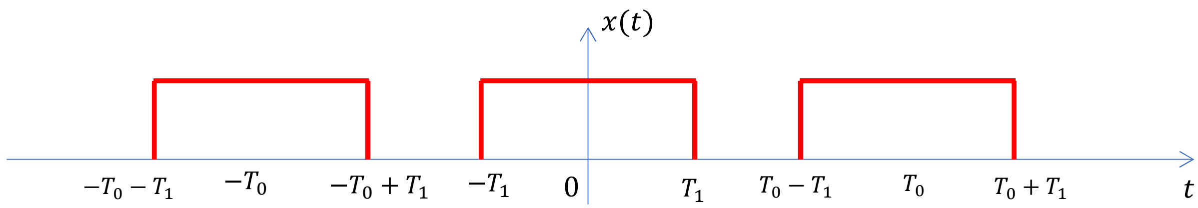
For a real signal:
$|D_n|$ is always even.
$\angle D_n$ is always odd.

Fourier Series

Summation of periodic signals with fundamental frequencies ω_1 and ω_2 is periodic only and only if $\frac{\omega_1}{\omega_2}$ is a rational number. For example $\frac{2\pi}{5\pi} = \frac{2}{5}$ is rational and $\frac{2\pi}{5} = \frac{2\pi}{5}$ is not rational!

Fourier Series

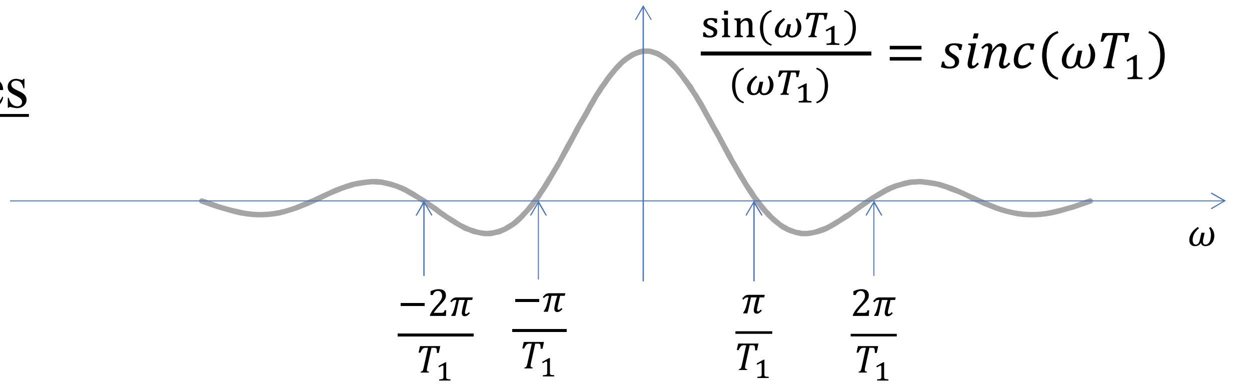
More complicated examples:



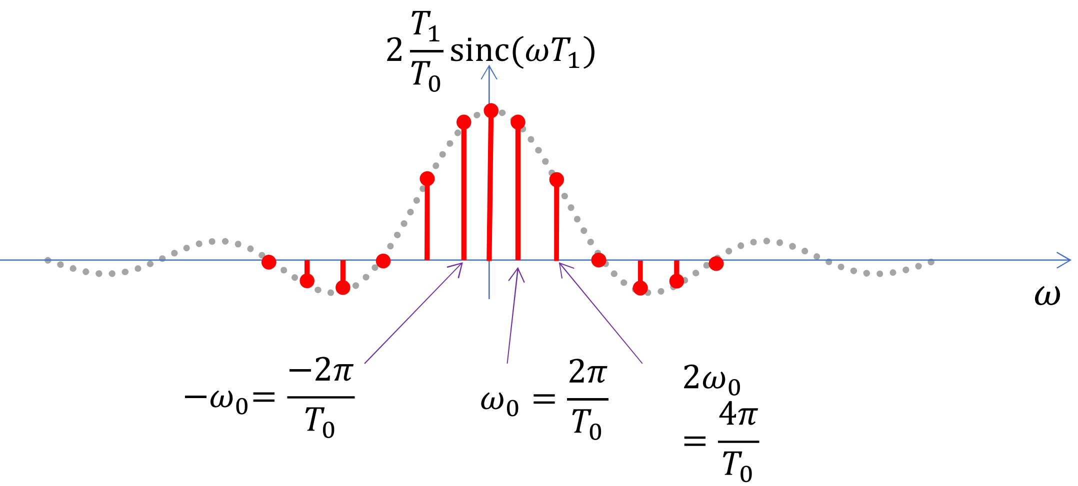
Find and plot the Fourier series of the above signal with period T_0.

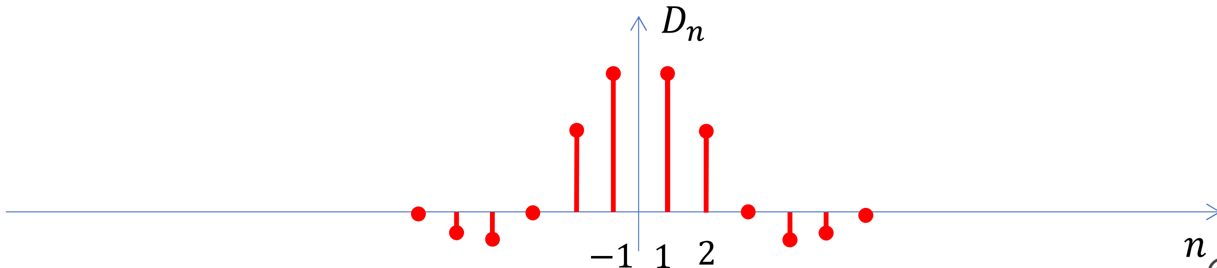
$$
\begin{aligned}
D_n &= \frac{1}{T_0}\int_{<T_0>} x(t)e^{-j\omega_0 nt}dt \\
&= \frac{1}{T_0}\int_{-T_1}^{T_1} 1.e^{-j\omega_0 nt}dt \\
&= -\frac{1}{T_0}\frac{e^{-j\omega_0 nt}}{j\omega_0 nt}\Big|_{-T_1}^{T_1} \\
&= \frac{e^{j\omega_0 nT_1} - e^{-j\omega_0 nT_1}}{j\omega_0 nT_0}
\end{aligned}
\qquad
\begin{aligned}
&= \frac{1}{\omega_0 nT_0}2sin(\omega_0 nT_1) \\
&= 2\frac{T_1}{T_0}\frac{sin(\omega_0 nT_1)}{\omega_0 nT_1} \\
&= 2\frac{T_1}{T_0}sinc(\omega_0 nT_1)
\end{aligned}
$$

Fourier Series



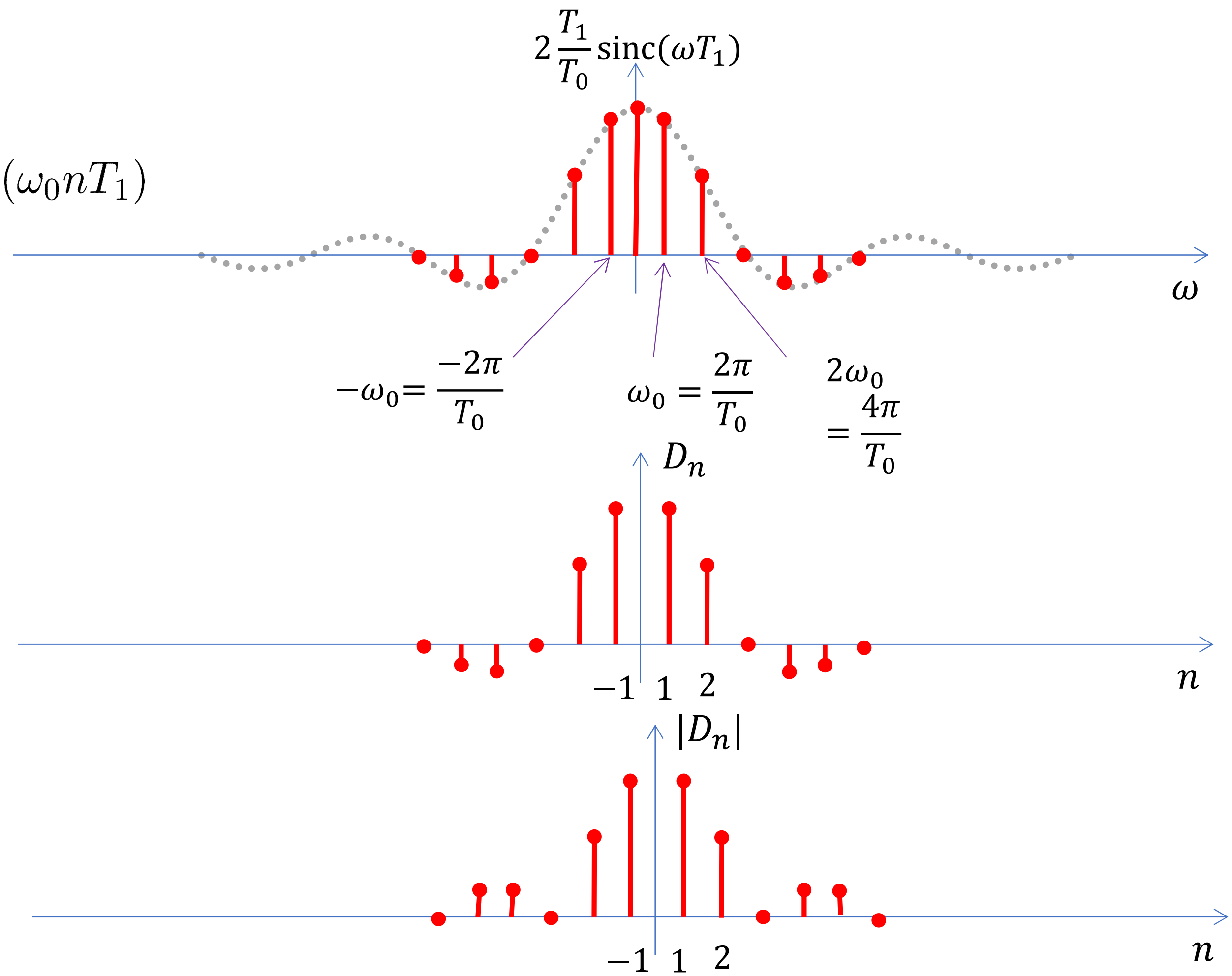
$$D_n = 2\frac{T_1}{T_0} sinc(\omega_0 n T_1)$$





Fourier Series

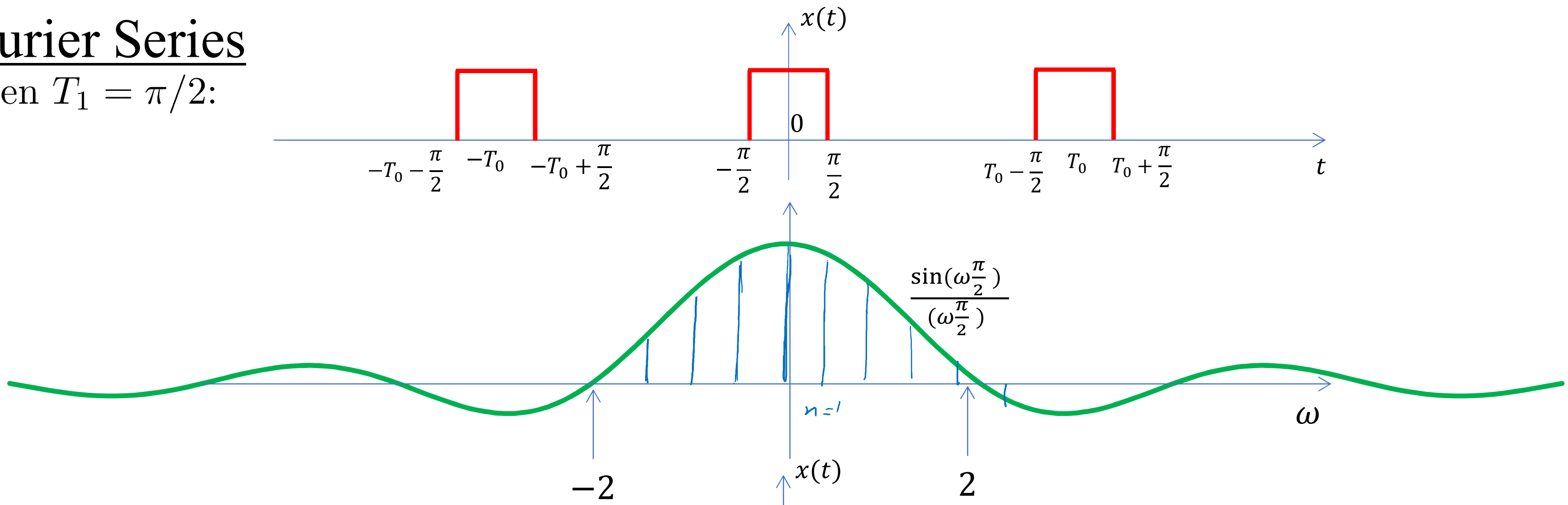
$$D_n = 2\frac{T_1}{T_0} sinc(\omega_0 n T_1)$$



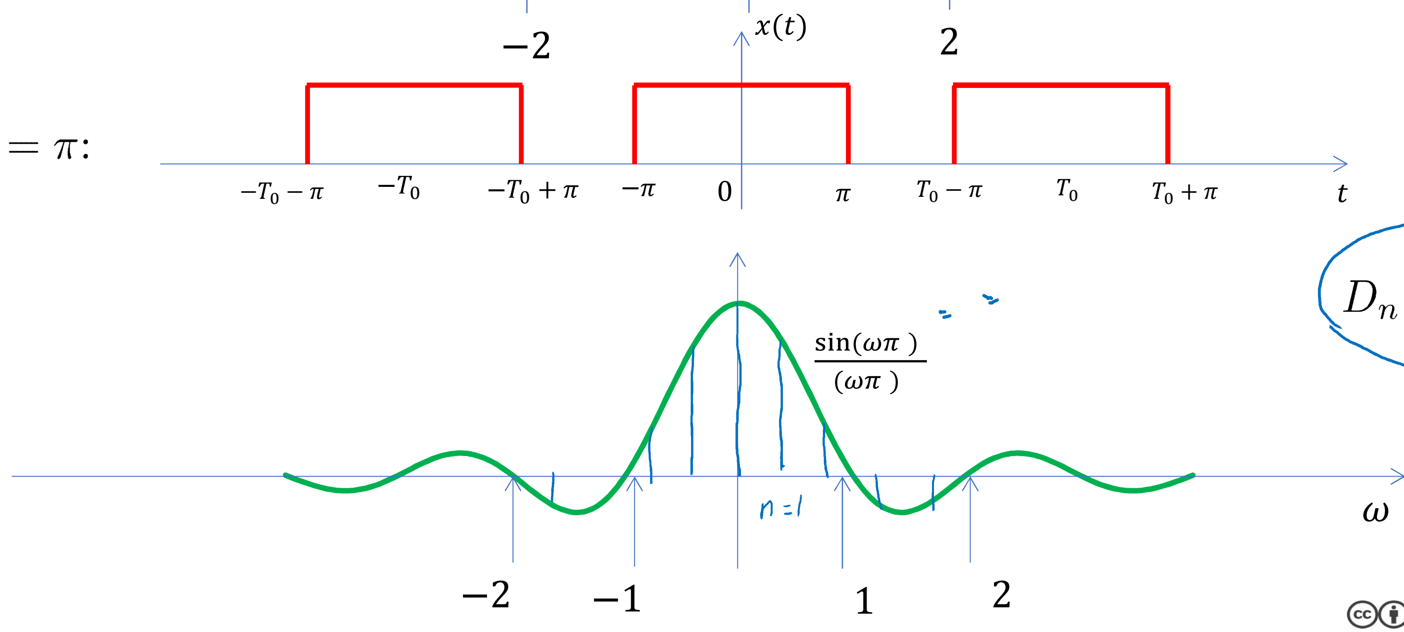
$\angle(D_n)$ is zero for positive values of D_n
and is π (or equivalently $-\pi$) for negative values of D_n

Fourier Series

When $T_1 = \pi/2$:



When $T_1 = \pi$:



$$D_n = 2\frac{T_1}{T_0}sinc(\omega_0 n T_1)$$

Trigonometric Fourier Series and Exponential Fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 nt} \qquad \text{(Exponential Fourier series)}$$

$$x(t) = D_0 + \underbrace{D_1 e^{j\omega_0 t} + D_{-1} e^{-j\omega_0 t}} + \underbrace{D_2 e^{j2\omega_0 t} + D_{-2} e^{-j2\omega_0 t}} + \underbrace{D_3 e^{j3\omega_0 t} + D_{-3} e^{-j3\omega_0 t}} + \cdots$$

$$x(t) = a_0 + \overbrace{a_1 \cos(\omega_0 t) + b_1 \sin(\omega_0 t)} + \overbrace{a_2 \cos(2\omega_0 t) + b_2 \sin(2\omega_0 t)} + \overbrace{a_3 \cos(3\omega_0 t) + b_3 \sin(3\omega_0 t)}$$

Euler formula

$$\overbrace{a_1\left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right) + b_1\left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right)}$$

$$\overbrace{\left(\frac{a_1}{2} + \frac{b_1}{2j}\right)e^{j\omega_0 t} + \left(\frac{a_1}{2} - \frac{b_1}{2j}\right)e^{-j\omega_0 t}}$$

$$\text{for} \quad n > 0 \quad \begin{cases} D_0 = a_0 \\ D_n = \frac{a_n}{2} + \frac{b_n}{2j} \\ D_{-n} = \frac{a_n}{2} - \frac{b_n}{2j} \end{cases} \overset{\text{or}}{\Longrightarrow} \begin{cases} D_n + D_{-n} = a_n \\ D_n - D_{-n} = \frac{b_n}{j} \end{cases}$$

Simple Example:

$$x(t) = cos(\omega_0 t)$$

Trigonometric FS: $a_0 = 0, a_1 = 1, a_2 = 0, ..., b_n = 0, \quad \forall n$

Exponential FS: $D_0 = 0, D_1 = \frac{1}{2}, D_{-1} = \frac{1}{2}$