Signals and Systems I

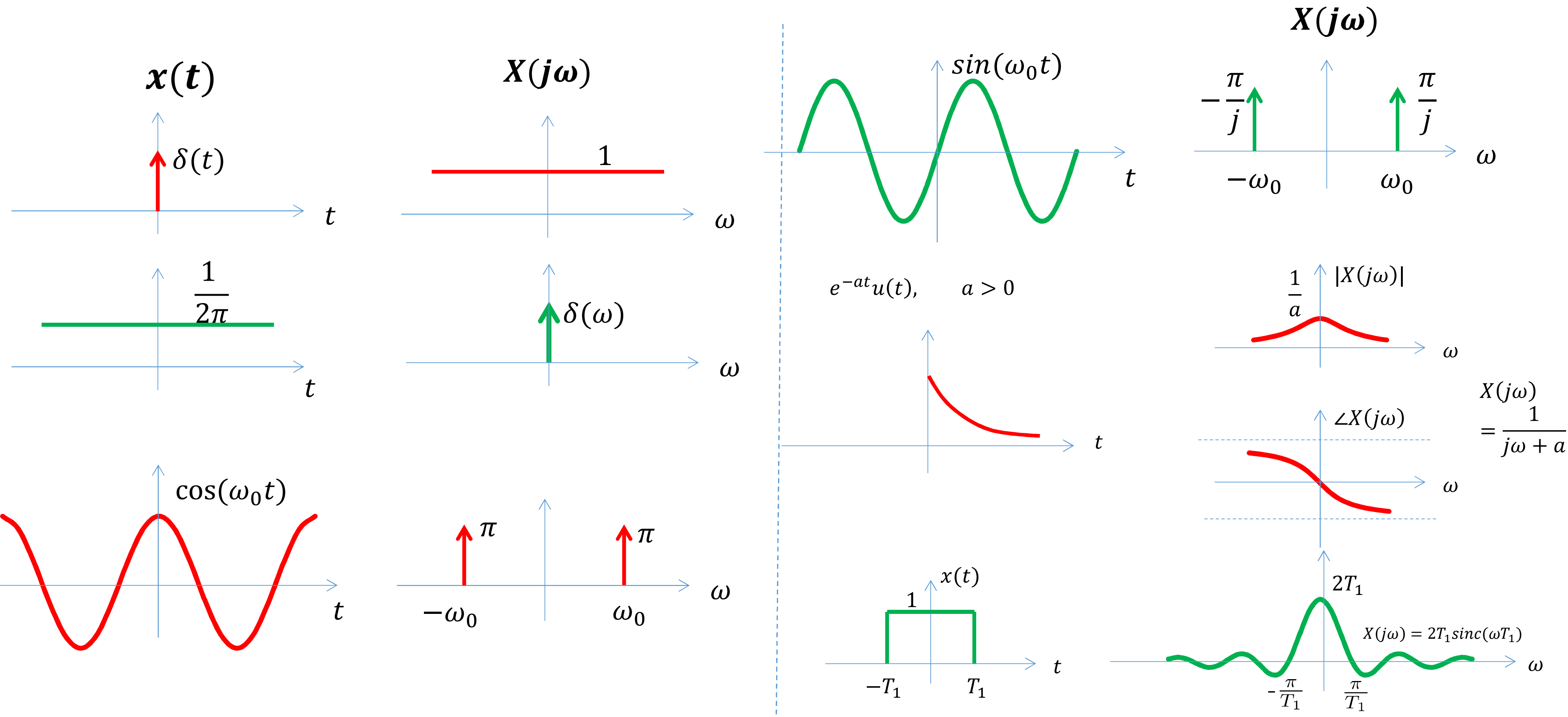
Topic 10

Synthesis Equation: $x(t) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$

Analysis Equation: $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

- More on calculation of FT and additional FT properties
- FT & LTI systems
- FT & LTIDE Systems
- Modulation & Demodulation
- Sampling

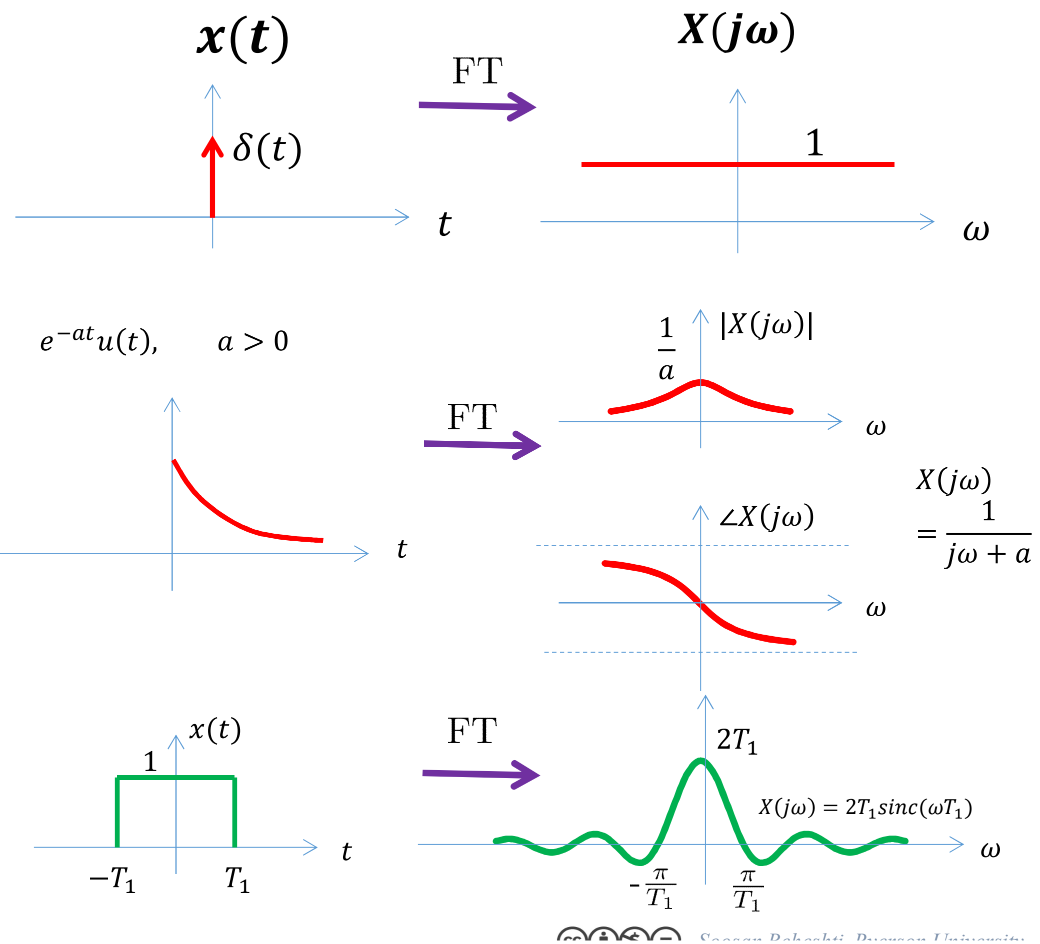
Fourier Transform of some Important Signals (review)



Fourier Transform of some Important Signals

Synthesis Equation: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$

Analysis Equation: $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$



The FTs can be calculated using the FT equation (The Analysis equation)

Fourier Transform of some Important Signals

Synthesis Equation: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$

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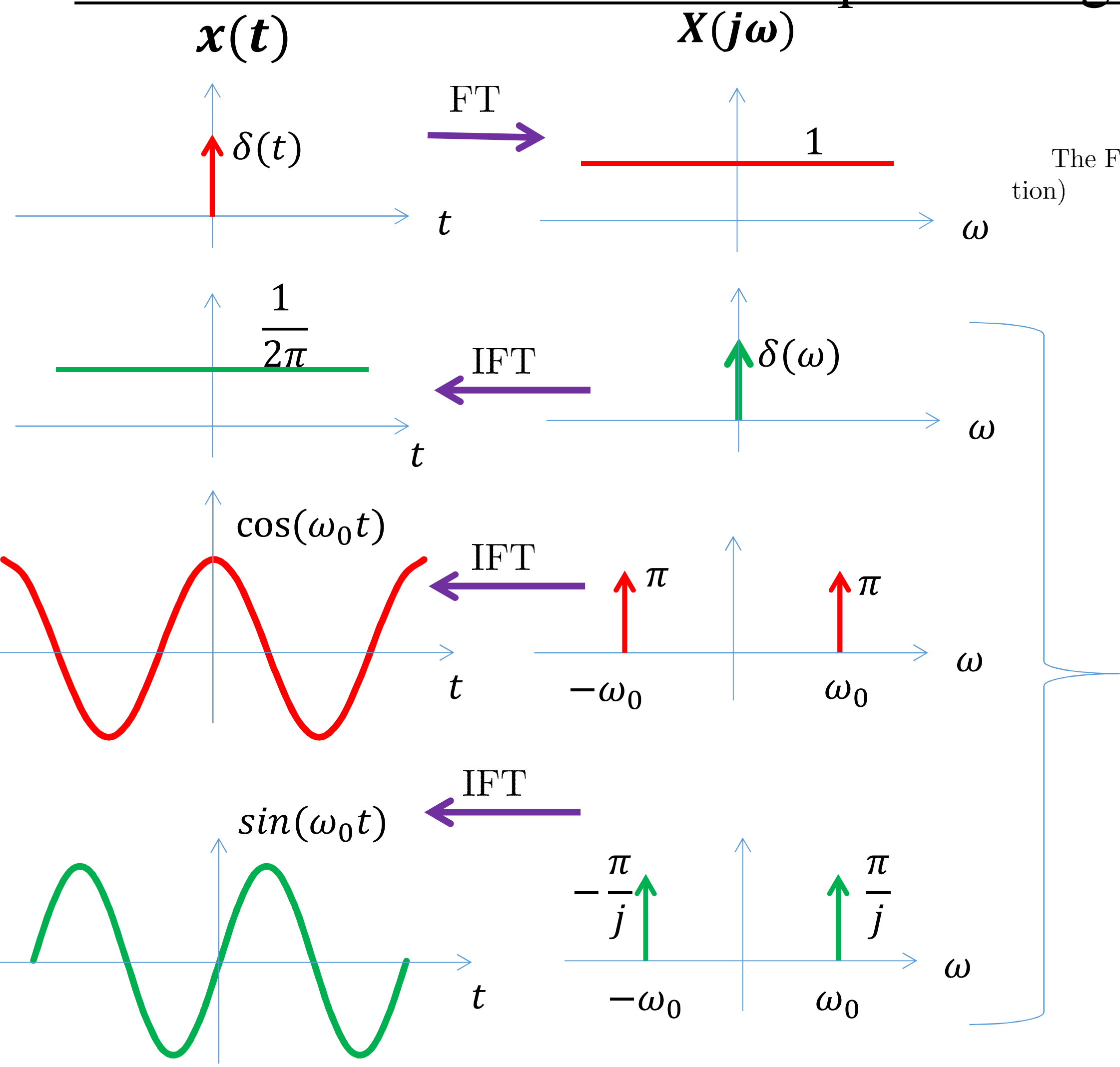


The FT can be easily calculated using the FT equation (The Analysis equation)

The inverse FT (IFT) can be easily calculated using:

1-The IFT equation (Synthesis equation)

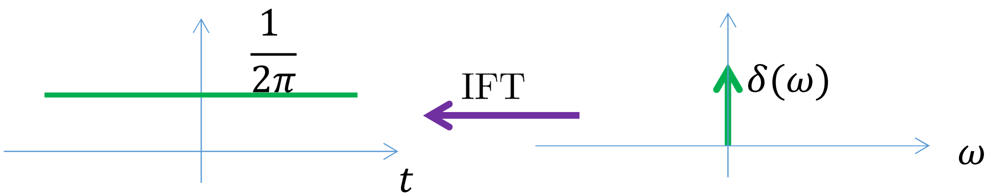
2- Duality property

The FT can be calculated

3- using the periodic property

4- or directly by using the synthesis equation! (not as convenient as the above methods)

Fourier Transform of a constant signal



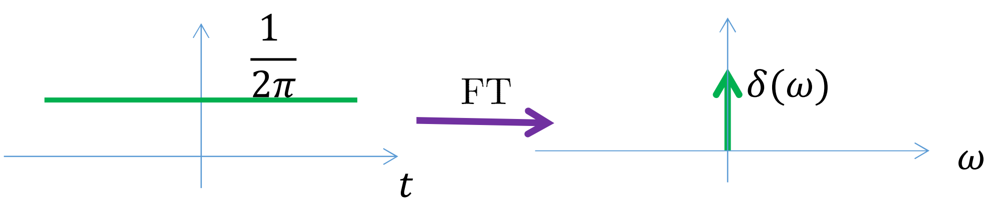
The inverse FT (IFT) can be easily calculated using:

1-The IFT equation (Synthesis equation)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)d\omega = \frac{1}{2\pi}$$

2- **Duality** property

$$\delta(t) \xrightarrow{\text{FT}} \Delta(j\omega) = 1 \xrightarrow{\text{FT}} 2\pi\delta(\omega)$$



The FT can be calculated

3- using the periodic property:

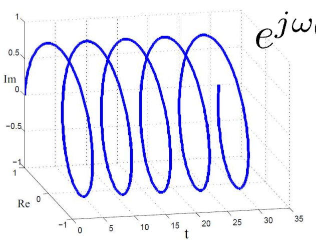
Constant signal is a pulse from $-T$ to T with period $2T$:

$D_0 = \frac{1}{2\pi}$ and the rest of the D_ns are zero. Therefore $X(j\omega) = 2\pi \times D_0\delta(\omega)$

4- or directly by using the synthesis equation! (not as convenient as the above methods):

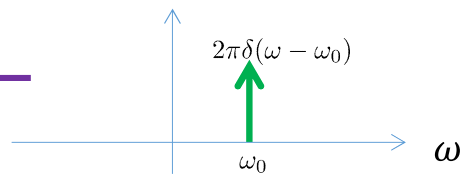
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2\pi}e^{-j\omega t}dt = \begin{cases} \delta(\omega) & \text{if } \omega = 0 \\ 0 & \text{if } \omega \neq 0 \text{ (why?)} \end{cases}$$

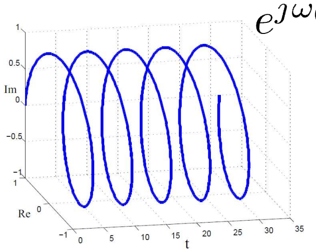
Fourier Transform of Periodic Spiral



$e^{j\omega_0 t}$

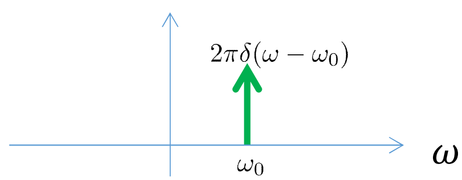
IFT





$e^{j\omega_0 t}$

FT



Orthogonality principle:

$$\int_{-\infty}^{\infty} e^{j\omega_0 t} e^{j\omega t} dt = \begin{cases} 0 & \text{if} \quad \omega_0 \neq \omega \\ 2\pi\delta(\omega-\omega_0) & \text{if} \quad \omega_0 = \omega \end{cases}$$

Remember:

$$e^{j\omega_0 t} \xrightarrow{\text{Fourier Transform}} 2\pi\delta(\omega-\omega_0)$$

The inverse FT (IFT) can be easily calculated using:

1- **Direct** use of IFT definition:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)e^{j\omega t}d\omega$$
$$= \int_{-\infty}^{\infty} \delta(\omega-\omega_0)e^{j\omega_0 t}d\omega = e^{j\omega_0 t}\underbrace{\int_{-\infty}^{\infty} \delta(\omega-\omega_0)d\omega}_{1} = e^{j\omega_0 t}$$

2- **Frequency Shift** property

$\delta(\omega) \xrightarrow{IFT} \frac{1}{2\pi}$ (constant signal)

Shifting the FT by ω is multiplication by $e^{j\omega_0 t}$ in time.

3- **Shift in time** property and **Duality** property

$x_1(t) = \delta(t+\omega_0) \xrightarrow{\text{FT}} X(j\omega) = e^{j\omega_0\omega}$

$x_2(t) = X(jt) = e^{j\omega_0 t} \xrightarrow{\text{FT}} 2\pi x_1(-\omega) = 2\pi\delta(\omega-\omega_0)$

The FT can be calculated

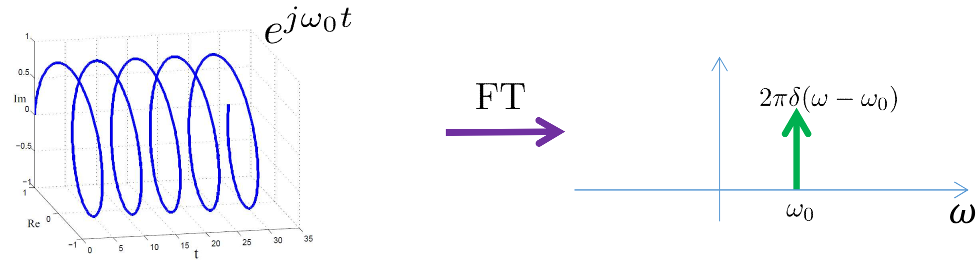
4- using the periodic property:

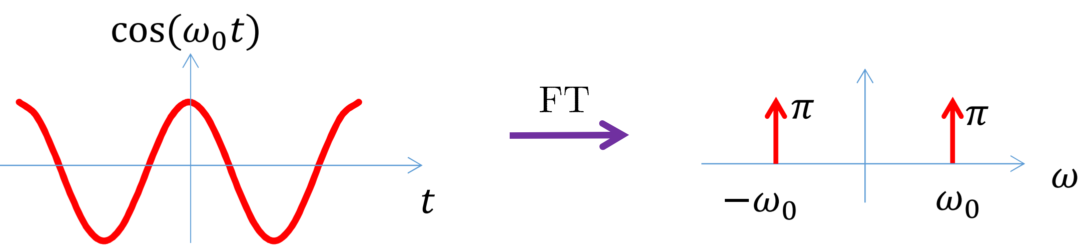
It is known that periodic spiral has only one nonzero D_n. If ω is positive $D_1 = 1$ and if it is negative $D_{-1} = 1$. Therefore $X(j\omega) = 2\pi \times D_1\delta(\omega-\omega_0)$ for positive ω_0 and $X(j\omega) = 2\pi \times D_{-1}\delta(\omega-\omega_0)$ for negative ω_0.

5- or directly by using the synthesis equation! (not as convenient as the above methods):

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{j\omega_0 t}e^{-j\omega t}dt = 2\pi\delta(\omega-\omega_0)$$

Fourier Transform of Cosine





Using either of five methods explained in the previous page, FT of cosine signal can be found through linear property of FT:

$$x(t) = \cos(t) = \tfrac{1}{2}e^{j\omega_0 t} + \tfrac{1}{2}e^{-j\omega_0 t} \xrightarrow{\text{FT}} X(j\omega) = \tfrac{1}{2}\text{FT}(e^{j\omega_0 t}) + \tfrac{1}{2}\text{FT}(e^{-j\omega_0 t})$$

$$= \tfrac{1}{2}2\pi\delta(\omega-\omega_0) + \tfrac{1}{2}2\pi\delta(\omega+\omega_0)$$

Fourier Transform Properties (Product of Signals)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Remember convolution in time

$$x_1(t) \xrightarrow{FT} X_1(j\omega)$$

$$x_2(t) \xrightarrow{FT} X_2(j\omega)$$

$$x_3(t) = x_1(t) * x_2(t) \xrightarrow{FT} X_3(j\omega) = X_1(j\omega) \times X_2(j\omega)$$

Convolution in time $\equiv$ Product in Frequency

9. FT of product of signals

$$x_1(t) \xrightarrow{FT} X_1(j\omega)$$

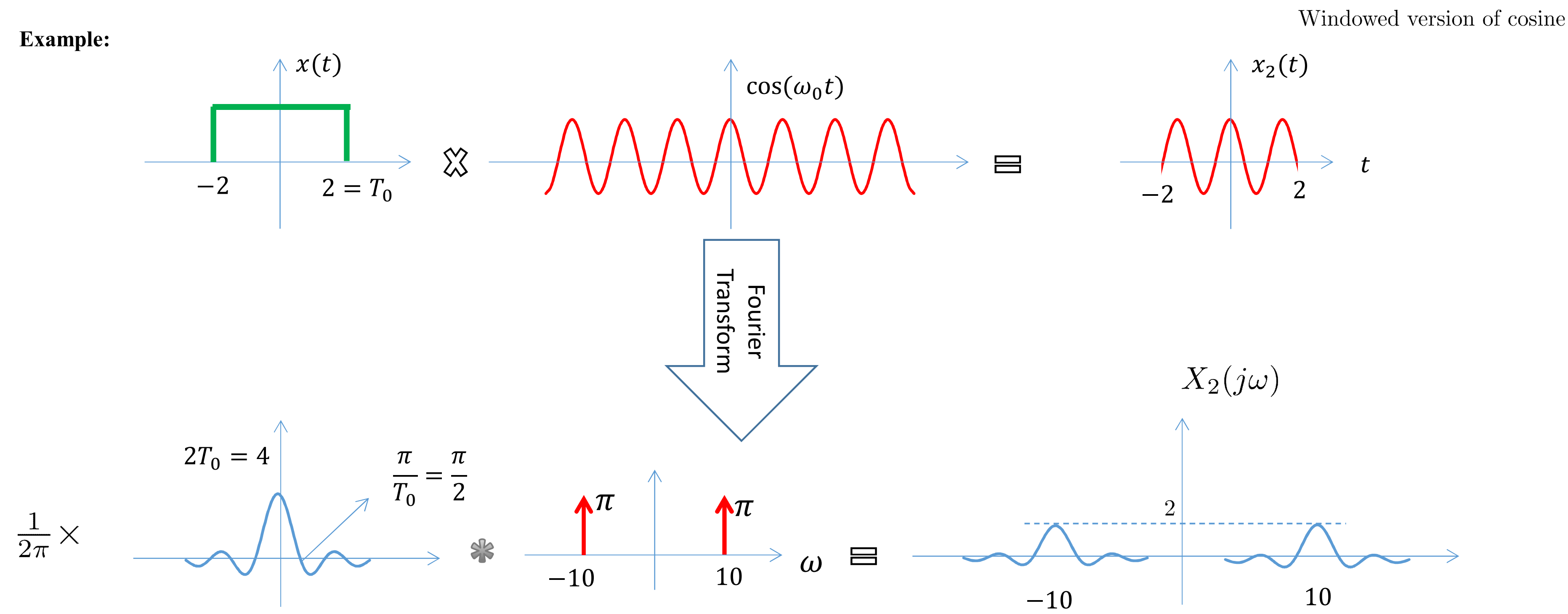
$$x_2(t) \xrightarrow{FT} X_2(j\omega)$$

$$x_3(t) = x_1(t) \times x_2(t) \xrightarrow{FT} X_3(j\omega) = \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$$

Product in time $\equiv$ Convolution in Frequency

Fourier Transform Properties (Product of Signals)

Example:



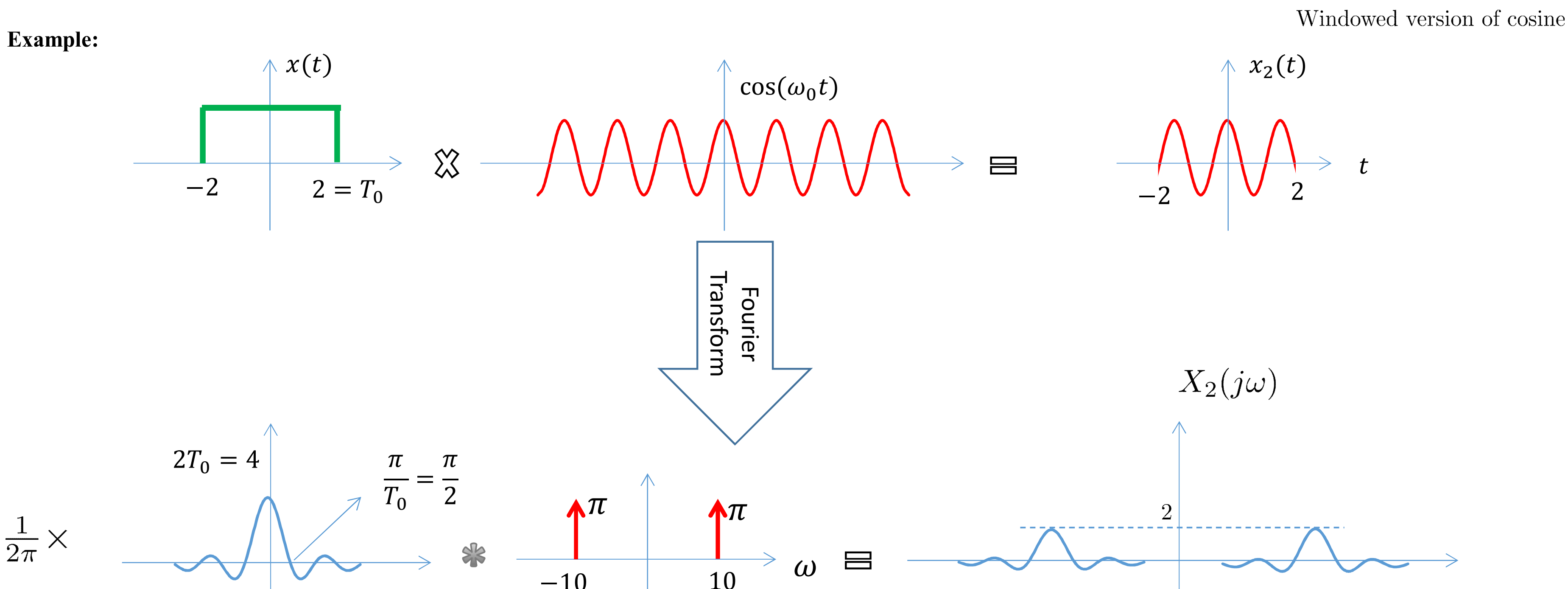
$$x_2(t) = cos(\omega_0 t) \times x(t), \qquad \omega_0 = 10$$

$$x(t) \times \underbrace{cos(10t+\theta)}_{\frac{1}{2}e^{j\theta}e^{j\omega_0 t}+\frac{1}{2}e^{-j\theta}e^{-j\omega_0 t}} \xrightarrow{FT} \frac{1}{2\pi}X(j\omega) * \left(\pi e^{j\theta}\delta(j\omega - j\omega_0) + \pi e^{-j\theta}\delta(j\omega + j\omega_0)\right)$$

$$= \quad \frac{1}{2}e^{j\theta}X\left(j(\omega-\omega_0)\right) + \frac{1}{2}e^{-j\theta}X\left(j(\omega+\omega_0)\right)$$

Fourier Transform Properties (Product of Signals)

Example:



Two interesting facts are shown by this simple example:

1- **Multiplication** of a signal (here a cosine) with a **pulse windows** the signal in time and equivalently **convolves** the FT of the signal with **sinc function**.

2- **Multiplication** of a signal (here a pulse) with **cosine Amplitude Modulates(AM)** the signal in time and equivalently the FT of the signal **shifts** to ω_o and $-\omega_0$

Fourier Transform Properties (Derivative and Integral)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

9. FT of derivative of a signal

$$x(t) \xrightarrow{FT} X(j\omega)$$

$$v(t) = x'(t) \xrightarrow{FT} V(j\omega) = j\omega X(j\omega)$$

proof: take the derivative of both sides of the synthesis equation:

$$\frac{d}{dt}x(t) = \frac{d}{dt}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d}{dt}X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)\frac{d}{dt}e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)j\omega e^{j\omega t}d\omega$$

10. FT of integral of a signal

$$x(t) \xrightarrow{FT} X(j\omega)$$

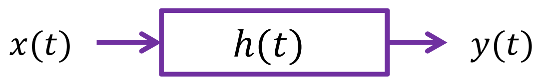
$$z(t) = \int_{-\infty}^{t} x(\tau)d\tau \xrightarrow{FT} Z(j\omega) = \frac{X(j\omega)}{j\omega}$$

Note that the integral property can be used only if $y(t) = \int_{-\infty}^{t} x(t)dt$ has a FT, i.e., the FT integral converges. For example it is known that $FT(\delta(t)) = 1$ and $u(t)$ is integral of $\delta(t)$ but $u(t)$ **doesn't have** a FT!

Fourier Transform & Periodic Signals

FT properties	Signal	FT
	$x(t)$	$X(j\omega)$
	$z(t)$	$Z(j\omega)$
Linearity	$ax(t) + bz(t)$	$aX(j\omega) + bZ(j\omega)$
Time shift	$x(t - T_0)$	$e^{-j\omega T_0} X(j\omega)$
Freq. shift	$e^{j\omega_0 t} x(t)$	$X(j(\omega - \omega_0))$
Scaling	$x(at)$	$\frac{1}{\lvert a \rvert} X(j\frac{\omega}{a})$
Duality	$X(jt)$	$2\pi x(-\omega)$
Complex Conj.	$x^*(t)$	$X^*(-j\omega)$ (so for real signals $\lvert X(j\omega\rvert$ is even and $\angle(X(j\omega)$ is odd
Convolution	$x(t) * z(t)$	$X(j\omega) \times Z(j\omega)$
Periodic signals	$x_p(t)$ with D_n coeffs	$X_p(j\omega) = \sum_n D_n 2\pi\delta(\omega - \omega_0 n)$
Product in time	$x(t) \times z(t)$	$\frac{1}{2\pi} X(j\omega) * Z(j\omega)$
Derivative	$x'(t)$	$j\omega X(j\omega)$
Integral	$\int_{-\infty}^{t} x(\tau) d\tau$	$\frac{X(j\omega)}{j\omega}$ only if the integral of FT conveges

Fourier Transform and LTI Systems



$$y(t) = h(t) * x(t), \quad Y(j\omega) = H(j\omega)X(j\omega),$$

Example:

$$H(j\omega) = \frac{1}{5 + 5j\omega}, \text{system is causal}$$

Reminder:

$$X_1(j\omega) = e^{-j4\omega} \longrightarrow \quad x_1(t) = \delta(t-4)$$

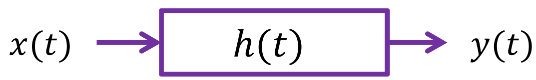
$$H(j\omega) = \frac{1}{5(j\omega+1)} \longrightarrow h(t) = \frac{1}{5}e^{-t}u(t)$$

$$X_2(j\omega) = \frac{1}{2+j\omega} \longrightarrow \quad x_2(t) = e^{-2t}u(t)$$

$$X_1(j\omega) = e^{-j4\omega} \quad \text{find} \quad y_1(t) = x_1(t) * h(t)$$

$$X_2(j\omega) = \frac{1}{2+j\omega} \quad \text{find} \quad y_2(t) = x_2(t) * h(t)$$

Fourier Transform and LTI Systems



$$y(t) = h(t) * x(t), \quad Y(j\omega) = H(j\omega)X(j\omega),$$

$$H(j\omega) = \frac{1}{5 + 5j\omega}$$

$$X_1(j\omega) = e^{-j4\omega} \quad \text{find} \quad y_1(t) = x_1(t) * h(t)$$

$$X_2(j\omega) = \frac{1}{2 + j\omega} \quad \text{find} \quad y_2(t) = x_2(t) * h(t)$$

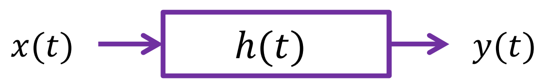
Method 1:

Find convolutions directly:

$$y_1(t) = \delta(t-4) * \left(\frac{1}{5}e^{-t}u(t)\right)$$

$$y_2(t) = e^{-2t}u(t) * \left(\frac{1}{5}e^{-t}u(t)\right)$$

Fourier Transform and LTI Systems



$$H(j\omega) = \frac{1}{5+5j\omega}$$

$$X_1(j\omega) = e^{-j4\omega} \quad \text{find} \quad y_1(t) = x_1(t) * h(t)$$

$$X_2(j\omega) = \frac{1}{2+j\omega} \quad \text{find} \quad y_2(t) = x_2(t) * h(t)$$

$$y(t) = h(t) * x(t), \quad Y(j\omega) = H(j\omega)X(j\omega),$$

Method 2:

Find $Y_1(j\omega)\&Y_2(j\omega)$ first and then take IFT of them to find $y_1(t)$ & $y_2(t)$.

$$Y_1(j\omega) = X_1(j\omega)H(j\omega) = e^{-j4\omega}\frac{1}{5+5j\omega} \xrightarrow{IFT} \delta(t-4) * h(t) = \frac{1}{5}e^{-(t-4)}u(t-4)$$

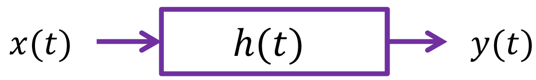
$$Y_2(j\omega) = X_2(j\omega)H(j\omega) = \frac{1}{2+j\omega} \times \frac{1}{5(1+j\omega)}$$

Use Partial Fraction Expansion and fine a and b to rewrite $Y(_2(j\omega)$ as

$$Y_2(j\omega) = \frac{a}{2+j\omega} + \frac{b}{1+j\omega}$$

Fourier Transform and LTI Systems

$$H(j\omega) = \frac{1}{5+5j\omega}$$



$X_1(j\omega) = e^{-j4\omega}$ find $y_1(t) = x_1(t) * h(t)$

$X_2(j\omega) = \frac{1}{2+j\omega}$ find $y_2(t) = x_2(t) * h(t)$

$$y(t) = h(t) * x(t), \quad Y(j\omega) = H(j\omega)X(j\omega),$$

Method 2:

Find $Y_1(j\omega)$&$Y_2(j\omega)$ first and then take IFT of them to find $y_1(t)$ & $y_2(t)$.

$$Y_1(j\omega) = X_1(j\omega)H(j\omega) = e^{-j4\omega}\frac{1}{5+5j\omega} \xrightarrow{IFT} \delta(t-4) * h(t) = \frac{1}{5}e^{-(t-4)}u(t-4)$$

$$Y_2(j\omega) = X_2(j\omega)H(j\omega) = \frac{1}{2+j\omega} \times \frac{1}{5(1+j\omega)}$$

Use Partial Fraction Expansion and fine a and b to rewrite $Y(_2(j\omega)$ as

$$Y_2(j\omega) = \frac{a}{2+j\omega} + \frac{b}{1+j\omega}$$

$$a = (2+j\omega)Y_2(j\omega)|_{j\omega=-2} = -\frac{1}{5}$$

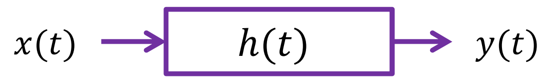
$$b = (1+j\omega)Y_2(j\omega)|_{j\omega=-1} = \frac{1}{5}$$

$$Y_2(j\omega) = \frac{\frac{1}{5}}{1+j\omega} + \frac{-\frac{1}{5}}{2+j\omega}$$

↓ IFT

Fourier Transform and LTI Systems

$$H(j\omega) = \frac{1}{5+5j\omega}$$



$$X_1(j\omega) = e^{-j4\omega} \quad \text{find} \quad y_1(t) = x_1(t) * h(t)$$

$$X_2(j\omega) = \frac{1}{2+j\omega} \quad \text{find} \quad y_2(t) = x_2(t) * h(t)$$

$$y(t) = h(t) * x(t), \quad Y(j\omega) = H(j\omega)X(j\omega),$$

Method 2:
Find $Y_1(j\omega)$&$Y_2(j\omega)$ first and then take IFT of them to find $y_1(t)$ & $y_2(t)$.

$$Y_1(j\omega) = X_1(j\omega)H(j\omega) = e^{-j4\omega}\frac{1}{5+5j\omega} \xrightarrow{IFT} \delta(t-4) * h(t) = \frac{1}{5}e^{-(t-4)}u(t-4)$$

$$Y_2(j\omega) = X_2(j\omega)H(j\omega) = \frac{1}{2+j\omega} \times \frac{1}{5(1+j\omega)}$$

Use Partial Fraction Expansion and fine a and b to rewrite $Y(_2(j\omega)$ as

$$Y_2(j\omega) = \frac{a}{2+j\omega} + \frac{b}{1+j\omega}$$

$$a = (2+j\omega)Y_2(j\omega)|_{j\omega=-2} = -\tfrac{1}{5}$$

$$b = (1+j\omega)Y_2(j\omega)|_{j\omega=-1} = \tfrac{1}{5}$$

$$Y_2(j\omega) = \frac{\frac{1}{5}}{1+j\omega} + \frac{-\frac{1}{5}}{2+j\omega}$$

IFT

$$y_2(t) = \tfrac{1}{5}e^{-t}u(t) - \tfrac{1}{5}e^{-2t}u(t)$$

Fourier Transform and LTIDE Systems

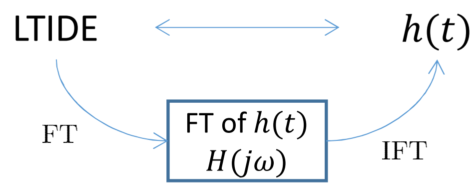
Find $h(t)$ for the following system:

$$y''(t) + ay'(t) + by(t) = cx'(t) + dx(t)$$

Useful properties:

$$e^{j\omega_0 t}X(j\omega) \xrightarrow{IFT} x(t - t_0)$$
$$j\omega X(j\omega) \xrightarrow{IFT} x'(t)$$
$$(j\omega)^2 X(j\omega) \xrightarrow{IFT} x''(t)$$

New approach:



$$FT\left(y''(t) + ay'(t) + by(t)\right) = FT\left(cx'(t) + dx(t)\right)$$
$$(j\omega)^2 Y(j\omega) + a(j\omega)Y(j\omega) + bY(j\omega) = c(j\omega)X(j\omega) + dX(j\omega)$$
$$Y(j\omega)\left((j\omega)^2 + a(j\omega) + b\right) = X(j\omega)\left(c(j\omega) + d\right)$$
$$Y(j\omega) = X(j\omega)\frac{c(j\omega) + d}{(j\omega)^2 + a(j\omega) + b}$$
$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{c(j\omega) + d}{(j\omega)^2 + a(j\omega) + b} \xrightarrow{IFT} h(t)$$

$h(t)$ is response to $x(t) = \delta(t)$ or equivalently is found by setting $X(j\omega) = 1$.

Fourier Transform and LTIDE Systems

Example: Find $h(t)$ for the following causal system by first finding $H(j\omega)$

$$y''(t) + 4y'(t) + 4y(t) = 2x'(t) + 2x(t)$$

Fourier Transform and LTIDE Systems

Example: Find $h(t)$ for the following causal system by first finding $H(j\omega)$

$$y''(t) + 4y'(t) + 4y(t) = 2x'(t) + 2x(t)$$

Solution:

$$\big((j\omega)^2 + 4j\omega + 4\big)\, Y(j\omega) = 2(j\omega + 1)X(j\omega)$$

$$Y(j\omega) = \underbrace{\frac{2(j\omega + 1)}{(j\omega)^2 + 4j\omega + 4}}_{H(j\omega)} X(j\omega)$$

$$H(j\omega) = \frac{2(j\omega + 1)}{(j\omega)^2 + 4j\omega + 4}$$

Two methods can be followed to find $h(t)$

Fourier Transform and LTIDE Systems

Method 1: Use Partial fractional expansion (PFE)

$$H(j\omega) = \frac{2(j\omega+1)}{(j\omega)^2+4j\omega+4}$$

$$H(j\omega) = \frac{A}{j\omega+2} + \frac{B}{(j\omega+2)^2}$$

$$B = H(j\omega)(j\omega+2)^2\Big|_{j\omega=-2} = 2(j\omega+1)\Big|_{j\omega=-2} = -2$$

$$A = \frac{d}{dj\omega}\left(H(j\omega)(j\omega+2)^2\right)\Big|_{j\omega=-2} = \frac{d}{dj\omega}2(j\omega+1) = 2$$

$$H(j\omega) = \frac{2}{j\omega+2} + \frac{-2}{(j\omega+2)^2}$$

$$\frac{2}{j\omega+2} \xrightarrow{IFT} 2e^{-2t}u(t)$$

$$\frac{-2}{(j\omega+2)^2} \xrightarrow{IFT} -2te^{-2t}u(t)$$

Reminder: $$\frac{1}{(j\omega+a)^2} \xrightarrow{IFT} te^{-at}u(t)$$

$$h(t) = (2e^{-2t} - 2te^{-2t})u(t)$$

Fourier Transform and LTIDE Systems

$$H(j\omega) = \frac{2(j\omega+1)}{(j\omega)^2+4j\omega+4}$$

Method 2:

$$H(j\omega) = \frac{2+2j\omega}{(j\omega+2)^2} = \frac{2}{(j\omega+2)^2} + \frac{2j\omega}{(j\omega+2)^2}$$

$$\frac{2}{(j\omega+2)^2} \xrightarrow{IFT} \quad 2te^{-2t}u(t)$$

$$j\omega.\frac{2}{(j\omega+2)^2} \xrightarrow{IFT} \quad \frac{d}{dt}\left(2te^{-2t}u(t)\right) = 2e^{-2t}u(t) - 4te^{-2t}u(t)$$

$$\begin{aligned} h(t) &= 2te^{-2t}u(t) + 2e^{-2t}u(t) - 4te^{-2t}u(t) \\ &= 2e^{-2t}u(t) - 2te^{-2t}u(t) \end{aligned}$$

Fourier Transform and LTIDE Systems

Example: Find the impulse response of the following causal system and also its output to $x(t) = e^{-6t}u(t)$ by using FT and $H(j\omega)$.

$$3y(t) + \frac{dy}{dt} = 8x(t) + 2\frac{dx}{dt}$$

Fourier Transform and LTIDE Systems

Example: Find the impulse response of the following causal system and also its output to $x(t) = e^{-6t}u(t)$ by using FT and $H(j\omega)$.

$$3y(t) + \frac{dy}{dt} = 8x(t) + 2\frac{dx}{dt}$$

Reminder: In this example $M = N$ and we have $b_0\delta(t)$ term in $h(t)$

$$H(j\omega) = \frac{8 + 2j\omega}{3 + j\omega} = 2 + \frac{2}{3 + j\omega}$$

$$h(t) = 2\delta(t) + 2e^{-3t}u(t)$$

$$Y(j\omega) = \frac{8 + 2j\omega}{3 + j\omega} \times \frac{1}{6 + j\omega} = \frac{a}{3 + j\omega} + \frac{b}{6 + j\omega}$$

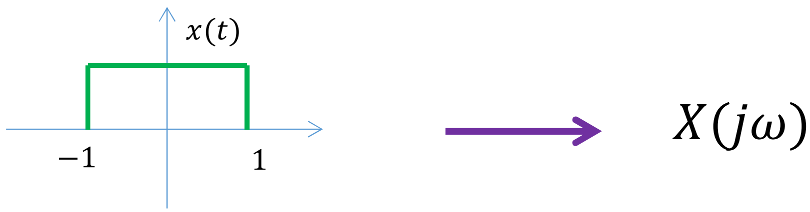
$$y(t) = \frac{2}{3}e^{-3t}u(t) + \frac{4}{3}e^{-6t}u(t)$$

Note: In using partial fraction expansion order of $j\omega$ in the numerator has to be less than the order of $j\omega$ in the denominatior.

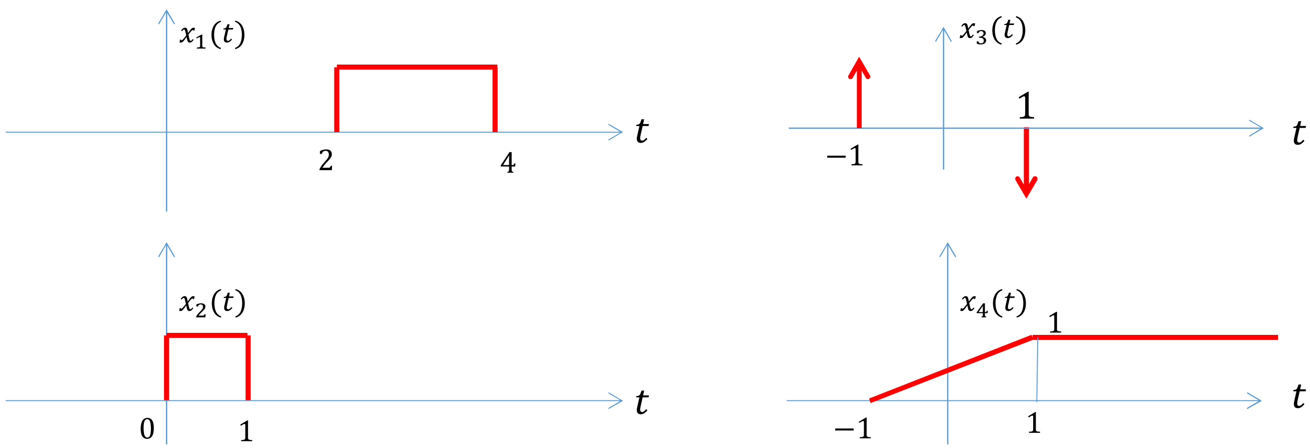
This method can be used for cases where M (order of highest derivate of input) is even greater than N (system order that is the order of highest derivative of output).

Using FT Properties

Consider the following signal

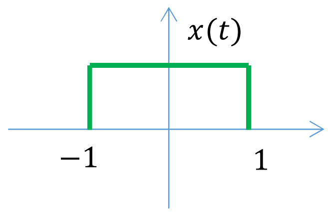


Write the FT of the following signals as a function of $X(j\omega)$. (Note this is not asking for finding the FT of the signals directly). Your answr must be only function of $X(j\omega)$.

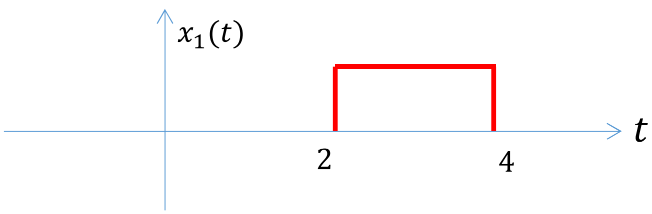


Using FT Properties

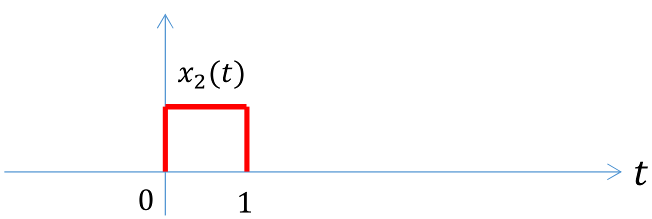
Consider the following signal



$\longrightarrow$ $X(j\omega)$



$$x_1(t) = x(t-3) \longrightarrow X_1(j\omega) = e^{-3j\omega} X(j\omega)$$



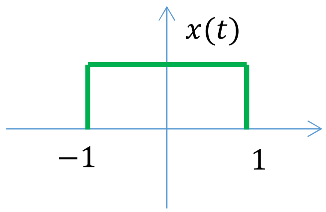
$$x_2(t) = x(2t-1)$$

$$z(t) = x(t-1) \longrightarrow Z(j\omega) = e^{-j\omega} X(j\omega)$$

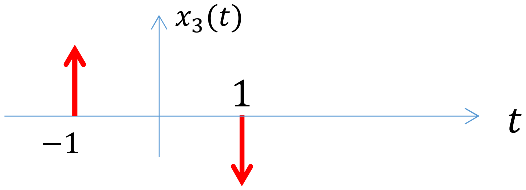
$$x_2(t) = z(2t) \longrightarrow X_2(j\omega) = \frac{1}{2} Z(\frac{j\omega}{2}) = \frac{1}{2} e^{-\frac{j\omega}{2}} X(\frac{j\omega}{2})$$

Using FT Properties

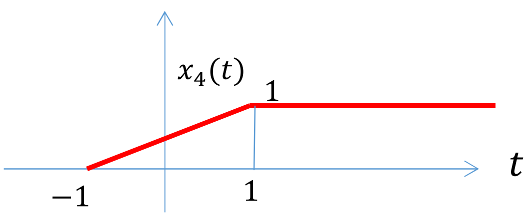
Consider the following signal



$\longrightarrow$ $X(j\omega)$



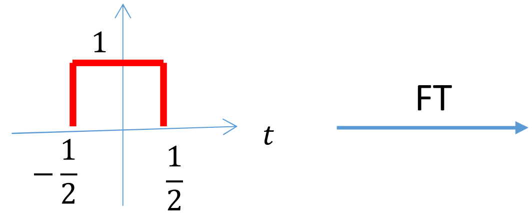
$$x_3(t) = \frac{d}{dt}x(t) \longrightarrow X_3(j\omega) = j\omega X(j\omega)$$



$$x_4(t) = \int_{-\infty}^{t} x(t)dt \longrightarrow X_4(j\omega) = \frac{1}{j\omega}X(j\omega)$$

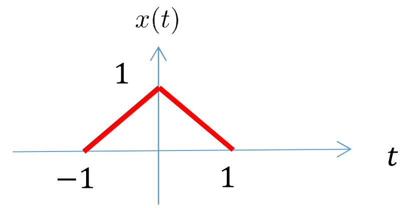
Using FT Properties

$v(t) = rect(t)$



$sinc(\frac{\omega}{2}) = \frac{\sin(\frac{\omega}{2})}{\frac{\omega}{2}}$

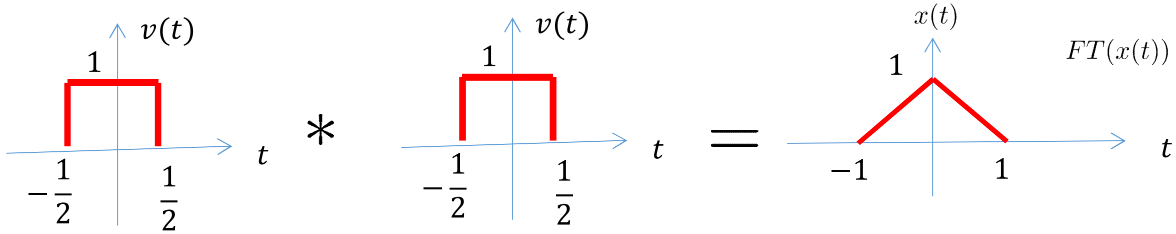
Find Fourier Transform of $x(t)$:



Method 1: Use the integral

$$X(j\omega) = \int_{-1}^{1} e^{-j\omega t}x(t)dt = \int_{-1}^{0}(t+1)e^{-j\omega t}dt + \int_{0}^{1}(-t+1)e^{-j\omega t}dt.$$

Method 2: Use FT properties Table:

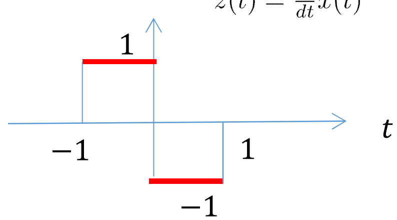


$$FT(x(t)) = FT(v(t) * v(t)) = V(j\omega) \times V(\omega) = sinc(\frac{\omega}{2}) \times sinc(\frac{\omega}{2}) = sinc^2(\frac{\omega}{2})$$

or:

Using Linearity and shift properties and then Euler's formula:

$z(t) = \frac{d}{dt}x(t)$



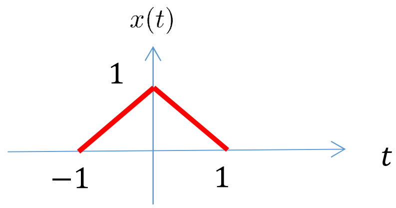
$$Ft(z(t)) = Ft(v(t+\frac{1}{2})) - Ft(v(t-\frac{1}{2})) = e^{j\omega}sinc(\frac{\omega}{2}) - e^{-j\omega}sinc(\frac{\omega}{2}) = (e^{j\omega} - e^{-j\omega})sinc(\frac{\omega}{2}) = 2j\sin(\frac{\omega}{2})sinc(\frac{\omega}{2})$$

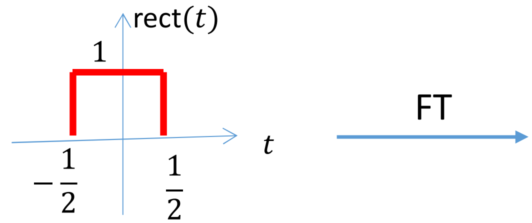
Using derivative property

$$Ft(x(t)) = \frac{1}{j\omega}FT(z(t) = \frac{1}{j\omega} \times 2j\sin(\frac{\omega}{2})sinc(\frac{\omega}{2}) = \frac{1}{\frac{\omega}{2}}\sin(\frac{\omega}{2})sinc(\frac{\omega}{2}) = sinc^2(\frac{\omega}{2})$$

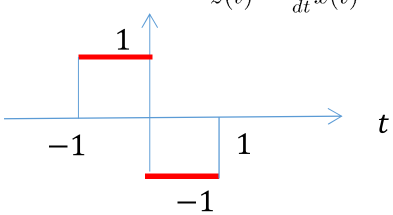
Using FT Properties

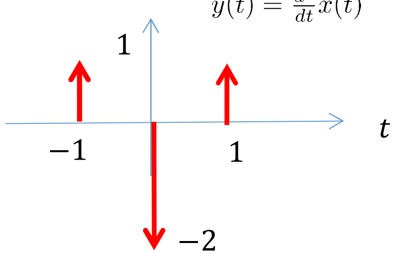
Find Fourier Transform of $x(t)$ by using its second derivatives:





$sinc(\frac{\omega}{2}) = \frac{\sin(\frac{\omega}{2})}{\frac{\omega}{2}}$





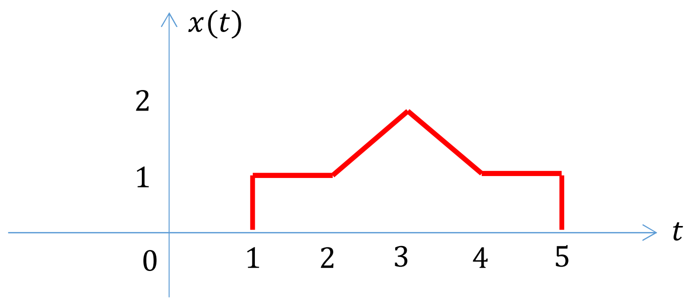
$$Ft(y(t)) = e^{j\omega} - 2 + e^{-j\omega} = 2\cos(j\omega) - 2$$

$$Ft(x(t)) = \frac{1}{j\omega}Ft(z(t)) = \frac{1}{j\omega} \times \frac{1}{j\omega}Ft(y(t)) = \frac{1}{-\omega^2}Ft(y(t))$$

$$= \frac{2(1-\cos(\omega))}{\omega^2} = \frac{4\sin(\frac{\omega}{2})}{\omega^2} = sinc^2(\frac{\omega}{2})$$

Using FT Properties

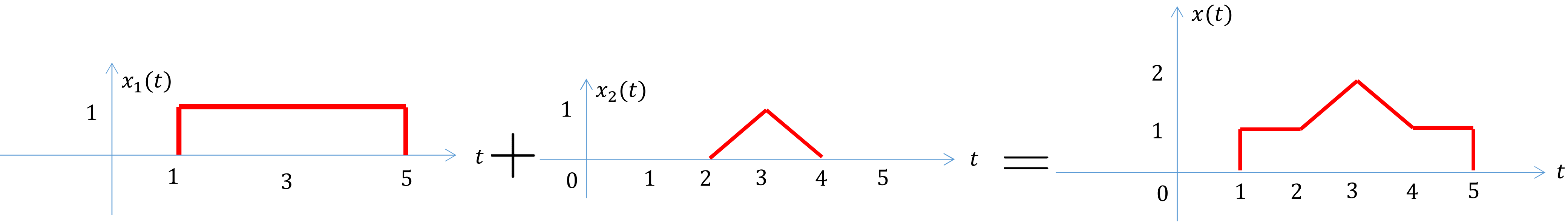
Example: Find the FT of the following signal:



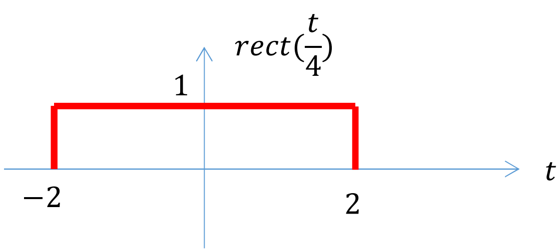
Method 1: Use the difficult integral $X(j\omega) = \int_1^5 e^{-j\omega t} x(t) dt$.

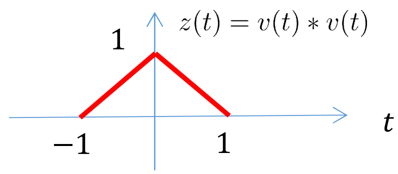
Method 2: Use FT properties Table

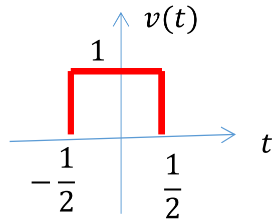
Using FT Properties



$$x(t) = x_1(t) + x_2(t)$$





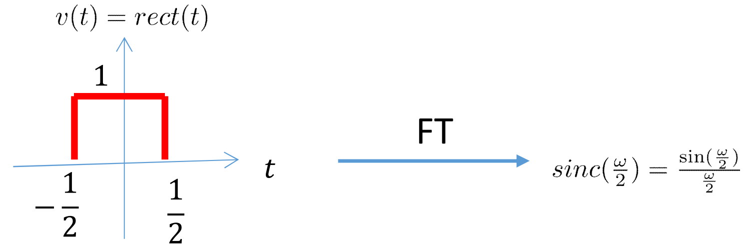


$$FT(x_1(t)) = FT(rect(\frac{t-3}{4}))$$
$$= e^{-3j\omega} FT(rect(\frac{t}{4}))$$
$$= e^{-3j\omega} 4sinc(\frac{4\omega}{2})$$

$$FT(x_2(t)) = FT(z(t-3)) = e^{-3j\omega} FT(z(t))$$
$$FT(z(t)) = FT(v(t) * v(t)) = V(j\omega) \times V(j\omega) = sinc^2(\frac{\omega}{2})$$
$$FT(x_2(t)) = e^{-3j\omega} sinc^2(\frac{\omega}{2})$$

$$FT(x(t)) = FT(x_1(t) + x_2(t)) = e^{-3j\omega} 4sinc(\frac{4\omega}{2}) + e^{-3j\omega} sinc^2(\frac{\omega}{2})$$

Using FT Properties



Or use derivative of $x(t)$ to find its FT:

$$z(t) = \delta(t-1) + v(t-\frac{5}{2}) - v(t-\frac{7}{2}) - \delta(t-5)$$

$$FT(z(t)) = e^{-j\omega} + e^{-\frac{5}{2}j\omega}FT(v(t)) - e^{-\frac{7}{2}j\omega}Ft(v(t)) - e^{-j5\omega}$$

$$= e^{-j\omega} - e^{-j5\omega} + e^{-3j\omega}(e^{\frac{1}{2}j\omega} - e^{-\frac{1}{2}j\omega})Ft(v(t))$$

$$= e^{-j3\omega}(e^{j2\omega} - e^{-j2\omega}) +$$

$$= 2je^{-3j\omega}\sin(2\omega) + 2je^{-3j\omega}\sin(\frac{\omega}{2})FT(v(t))$$

$$= 2je^{-j3\omega}(\sin(2\omega) + \sin(\frac{\omega}{2})sinc(\frac{\omega}{2}))$$

$$FT(x(t)) = \frac{1}{j\omega}FT(z(t)$$

$$= e^{-j3\omega}(\frac{4}{2\omega}\sin(2\omega) + \frac{1}{\frac{\omega}{2}}\sin(\frac{\omega}{2})sinc(\frac{\omega}{2}))$$

$$= e^{-j3\omega}(4sinc(2\omega) + sinc^2(\frac{\omega}{2}))$$

alternatively we can first work on shifted version of $z(t)$ that is symmetric. Compare the steps:

$$y(t) = z(t+3)$$

$$y(t) = \delta(t+2) + v(t+\frac{1}{2}) - v(t-\frac{1}{2}) - \delta(t-2)$$

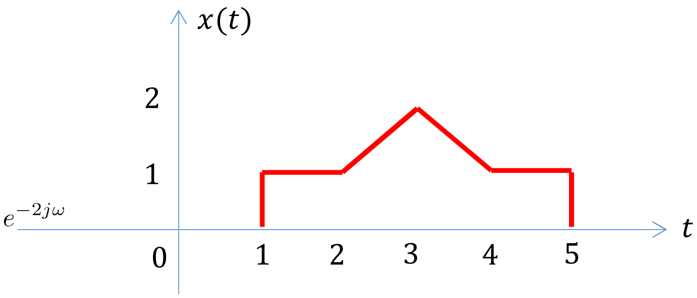
$$FT(y(t)) = e^{2j\omega} + e^{+\frac{1}{2}j\omega}FT(v(t)) - e^{-\frac{1}{2}j\omega}Ft(v(t)) - e^{-2j\omega}$$

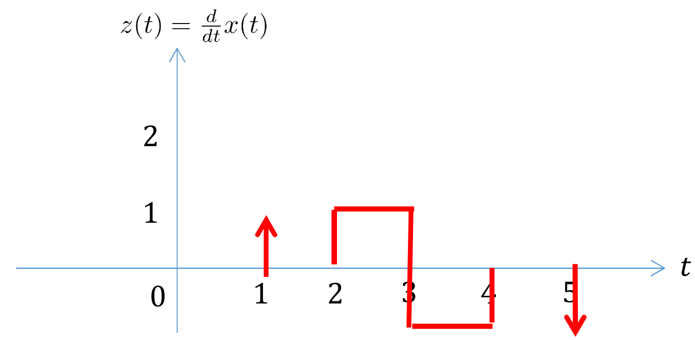
$$= e^{2j\omega} - e^{-2j\omega} + (e^{\frac{1}{2}j\omega} - e^{-\frac{1}{2}j\omega})Ft(v(t))$$

$$= 2j\sin(2\omega) + \sin(\frac{\omega}{2})FT(v(t))$$

$$= 2j\sin(2\omega) + \sin(\frac{\omega}{2})sinc(\frac{\omega}{2})$$

$$FT(z(t)) = e^{-j3\omega}FT(y(t))$$





Reminder: Fourier Transform Properties (Frequency Shift)

Fourier Transform of causal part of real part of an exponential signal ($\sigma > 0$):

$$x(t) = e^{-\sigma t}\cos(\omega_0 t)u(t)$$

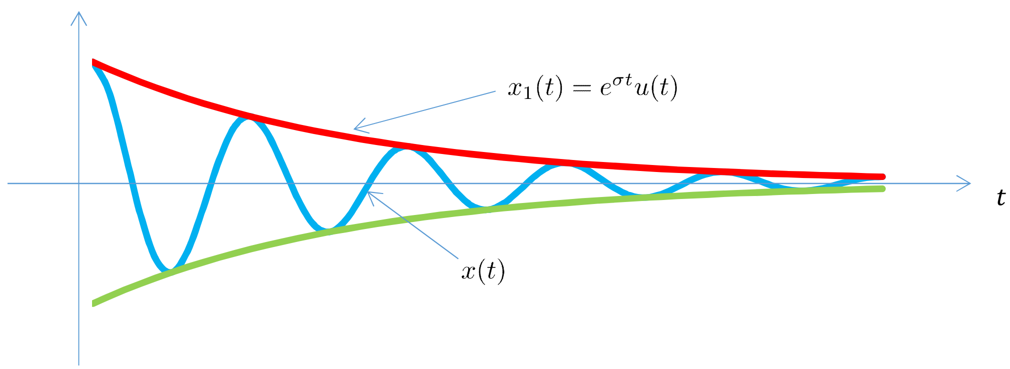
$$x_1(t) = e^{-\sigma t}u(t) \qquad FT(x_1(t)) = X_1(j\omega) = \frac{1}{j\omega + \sigma}$$

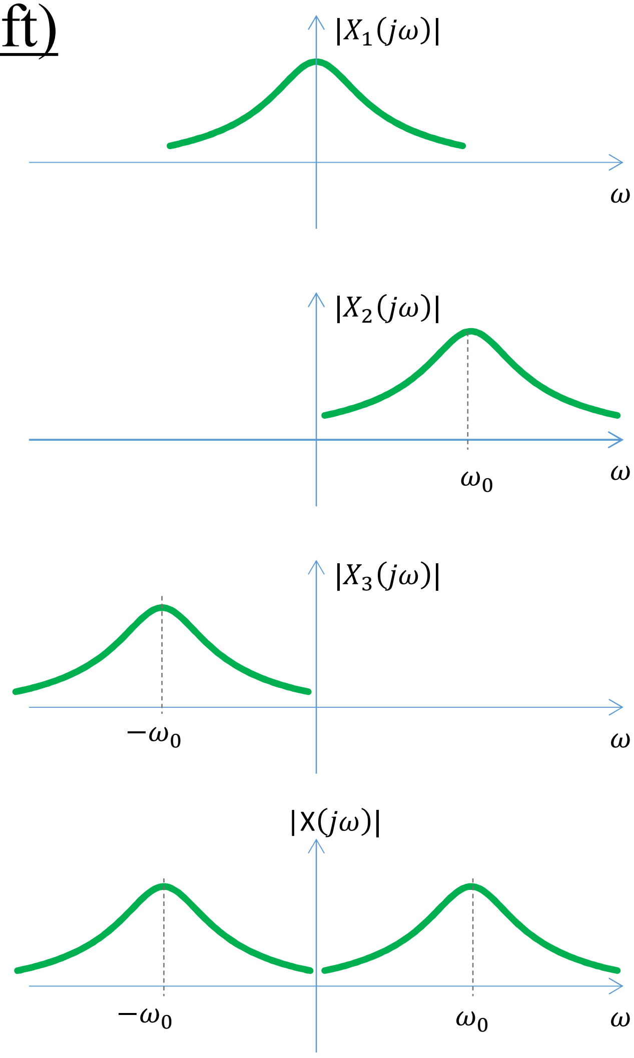
$$x(t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}x_1(t)$$

$$FT(x(t)) = \frac{1}{2}(X_1(j\omega - \omega_0) + X_1(j\omega + \omega_0))$$

$$= \frac{1}{2}(\frac{1}{j(\omega - \omega_0) + \sigma} + \frac{1}{j(\omega + \omega_0) + \sigma} = \frac{1}{2}(\frac{1}{j\omega + \sigma - j\omega_0} + \frac{1}{j\omega + \sigma + j\omega_0})$$

$$= \frac{1}{2}\frac{j\omega + \sigma - j\omega_0 + j\omega + \sigma + j\omega_0}{(j\omega + \sigma)^2 - (j\omega_0)^2} = \frac{j\omega + \sigma}{(j\omega + \sigma)^2 + \omega_0^2}$$





Using Fourier Transform Properties

Fourier Transform of causal part of an exponential signal $\sigma > 0$:

$$x(t) = e^{-st}u(t) = e^{-(\sigma+j\omega_0)t}u(t) \qquad X(j\omega) = \frac{1}{s+j\omega}$$

$$FT(e^{-\sigma t}\cos(\omega_0 t)u(t)) = \frac{1}{2}(FT(e^{(-\sigma+j\omega_0)t}u(t) + FT(e^{(-\sigma-j\omega_0)t}u(t))$$

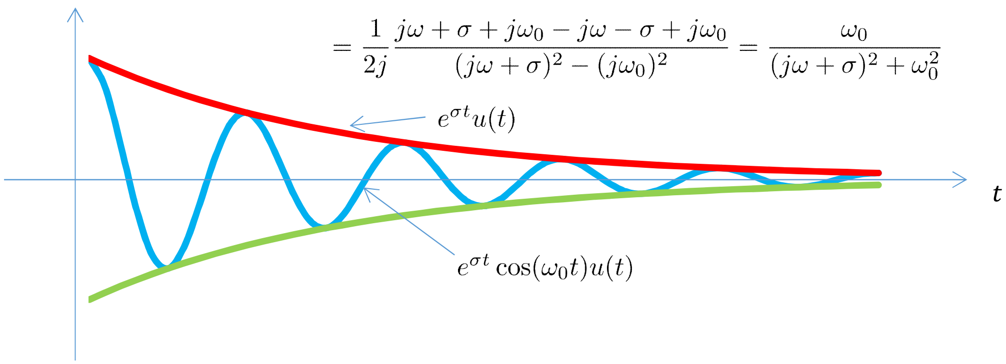
$$= \frac{1}{2}(\frac{1}{\sigma - j\omega_0 + j\omega} + \frac{1}{\sigma + j\omega_0 + j\omega})$$

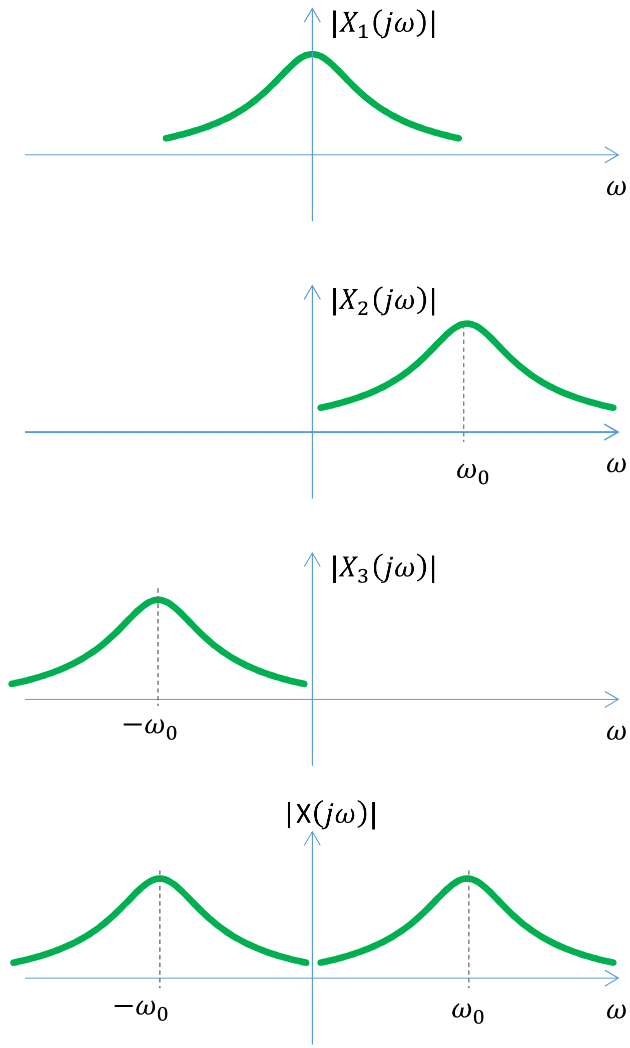
$$= \frac{1}{2}\frac{j\omega + \sigma + j\omega_0 + j\omega + \sigma - j\omega_0}{(j\omega+\sigma)^2 - (j\omega_0)^2} = \frac{j\omega + \sigma}{(j\omega+\sigma)^2 + \omega_0^2}$$

$$FT(e^{-\sigma t}\sin(\omega_0 t)u(t)) = \frac{1}{2j}(FT(e^{(-\sigma+j\omega_0)t}u(t) - FT(e^{(-\sigma-j\omega_0)t}u(t))$$

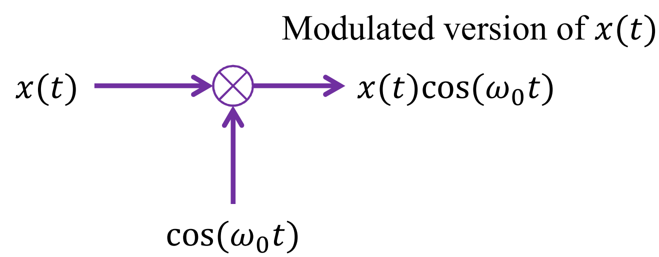
$$= \frac{1}{2j}(\frac{1}{\sigma - j\omega_0 + j\omega} - \frac{1}{\sigma + j\omega_0 + j\omega})$$

$$= \frac{1}{2j}\frac{j\omega + \sigma + j\omega_0 - j\omega - \sigma + j\omega_0}{(j\omega+\sigma)^2 - (j\omega_0)^2} = \frac{\omega_0}{(j\omega+\sigma)^2 + \omega_0^2}$$

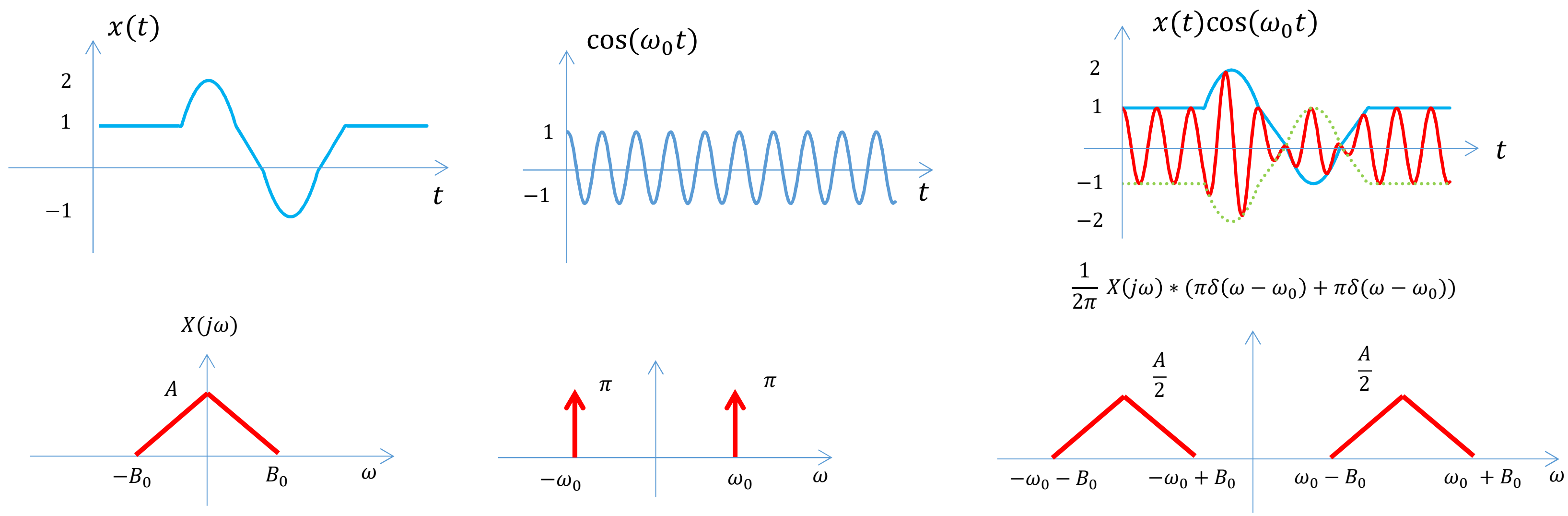




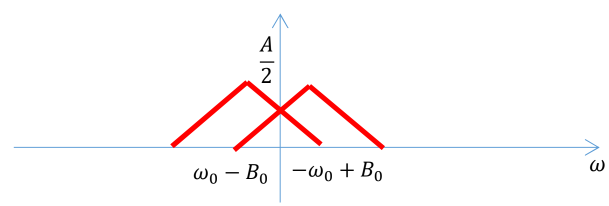
Modulation



$$x_1(t)x_2(t) \xrightarrow{FT} \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$$

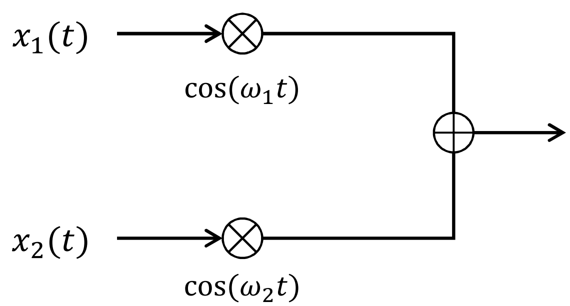


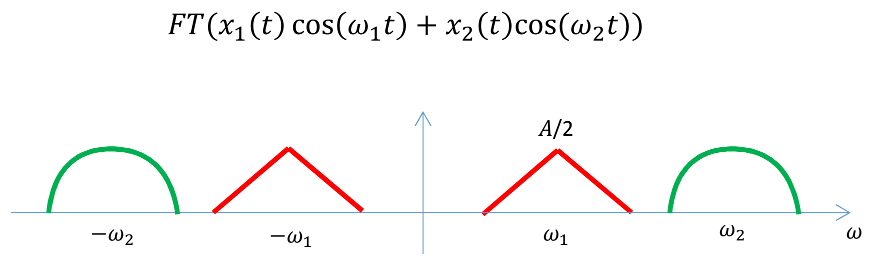
If $\omega_0 < B_0$
Aliasing

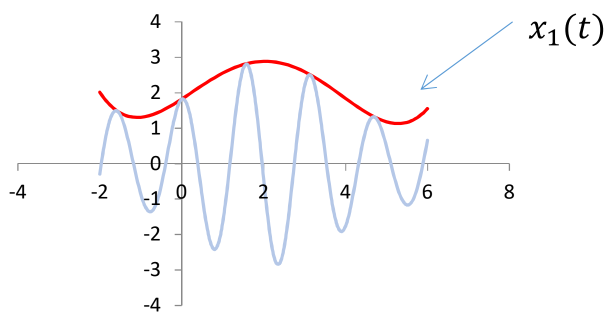


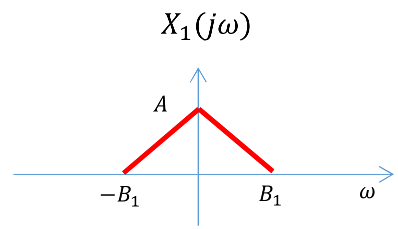
Modulation

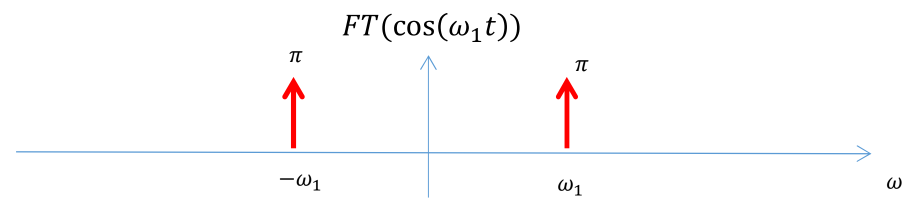
Two signals over the same channel

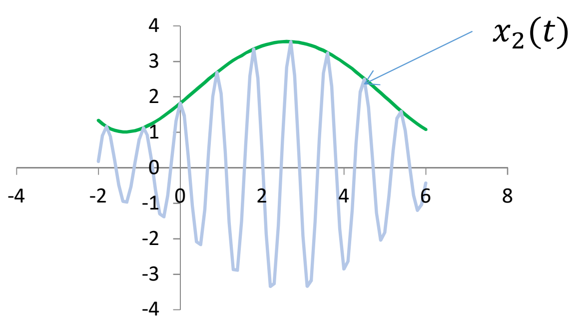


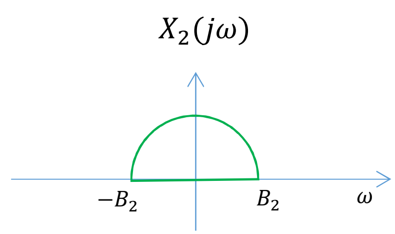


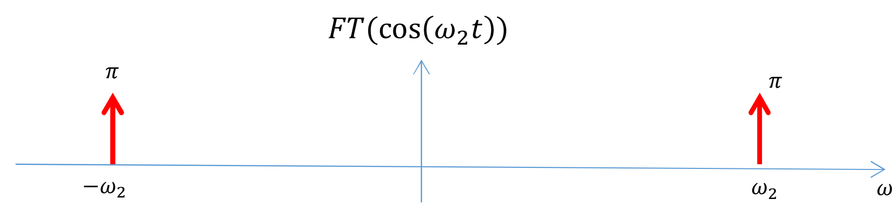








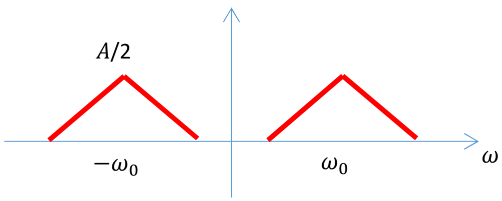




What are the conditions so that shapes of $X_1(j\omega)$ & $X_2(j\omega)$ are preserved?

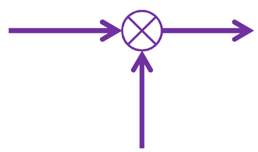
Demodulation

At the receiver multiply the signal to the same $cos(\omega_0 t)$



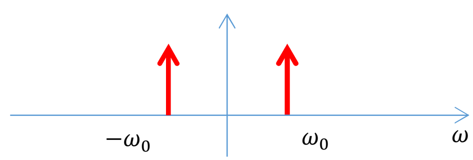
$y(t) = x(t)\cos(\omega_0 t)$

Modulated

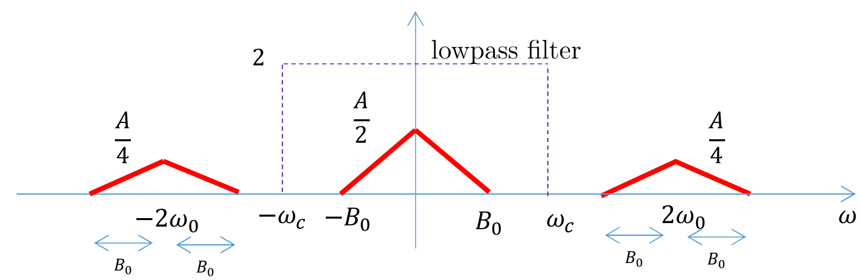


$r(t) = x(t)\cos(\omega_0 t)\cos(\omega_0 t)$

$\cos(\omega_0 t)$



$R(j\omega)$



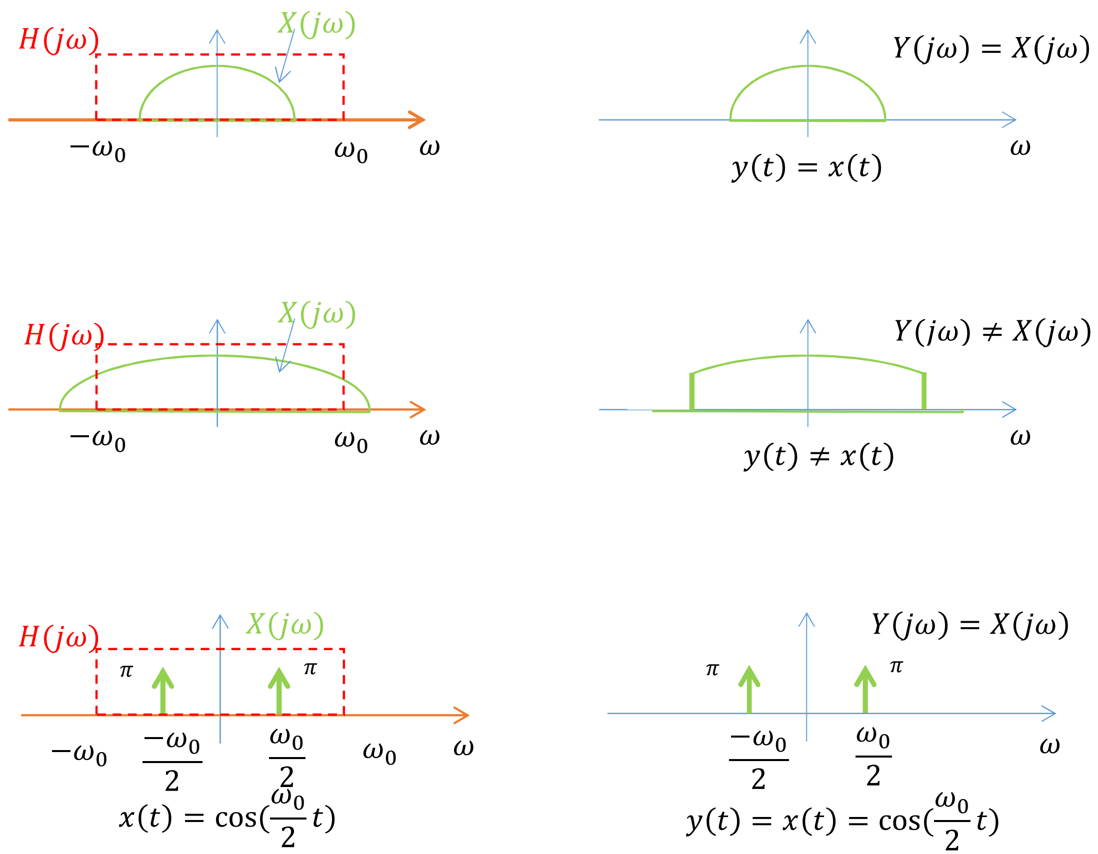
$$r(t) = x(t)\cos^2(\omega_0 t) = x(t)(\frac{1+\cos(2\omega_0 t)}{2}) = \frac{1}{2}x(t) + \frac{1}{2}\cos(2\omega_0 t)x(t)$$

$$x_1(t)x_2(t) \xrightarrow{FT} \frac{1}{2\pi}X_1(j\omega) * X_2(j\omega)$$

$$R(j\omega) = \frac{1}{2}X(j\omega) + \frac{1}{4}(X(j\omega - 2\omega_0) + X(j\omega + 2\omega_0))$$

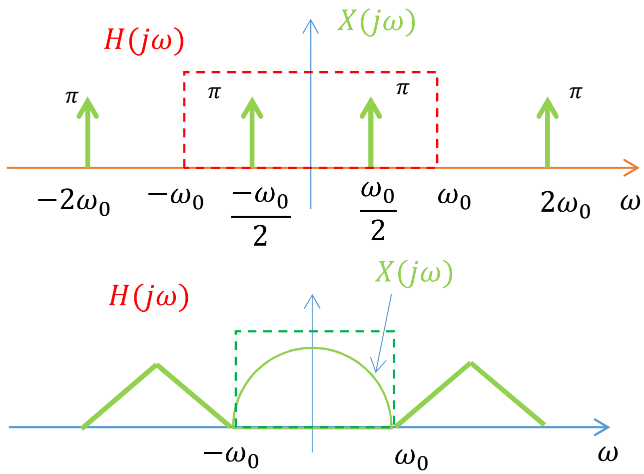
Lowpass Filter and signal recovery

$$y(t) = x(t) * h(t) \qquad Y(j\omega) = X(j\omega)H(j\omega)$$

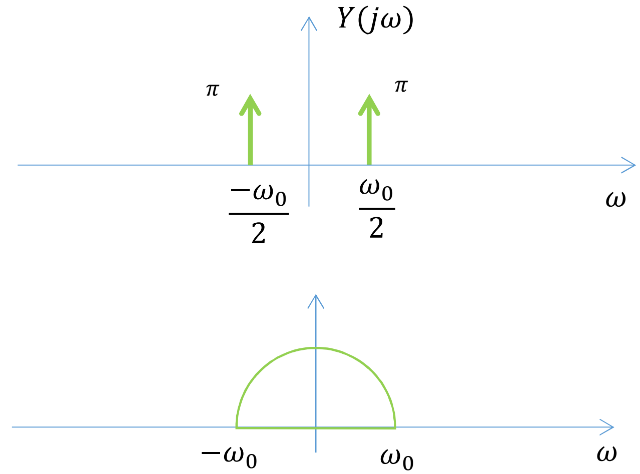


Demodulation

$$x(t) = \cos(\frac{\omega_0}{2}\, t) + \cos(2\omega_0)$$

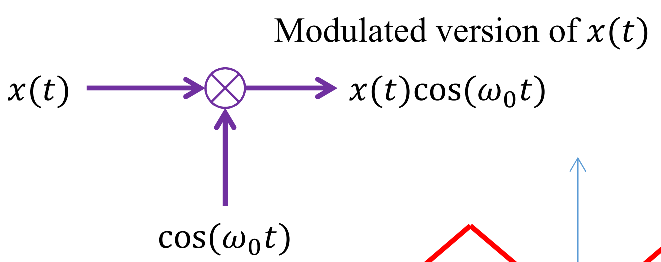


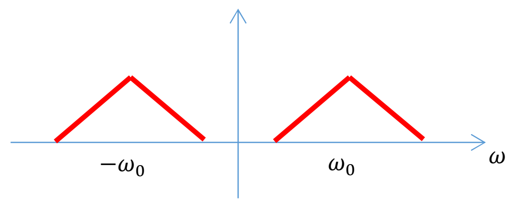
$$y(t) = \cos(\frac{\omega_0}{2}\, t)$$

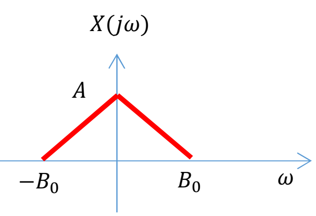


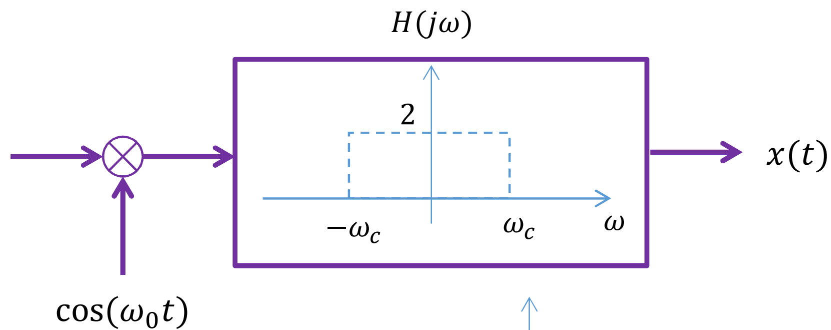
Modulation and Demodulation

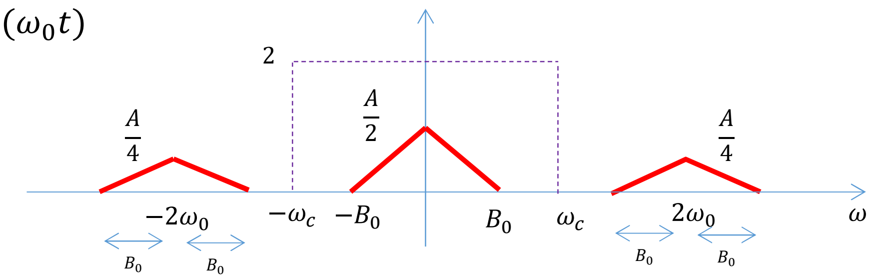
Next pass the received signal through a low pass filter!



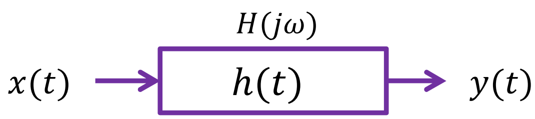




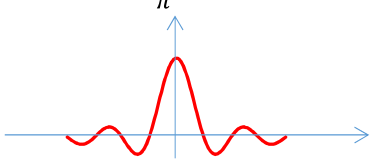


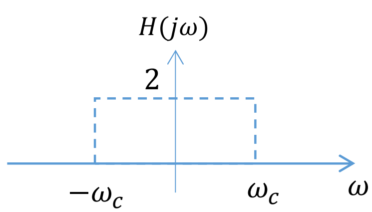


Low-pass filter



$h(t) = 2\frac{\omega_c}{\pi} sinc(\omega_c t)$

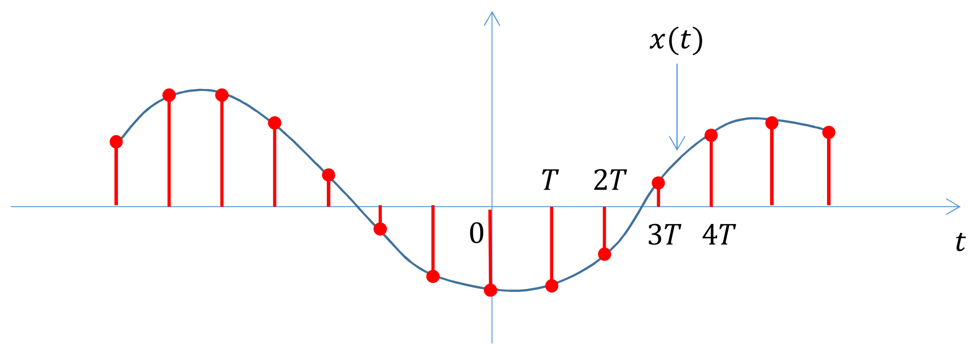


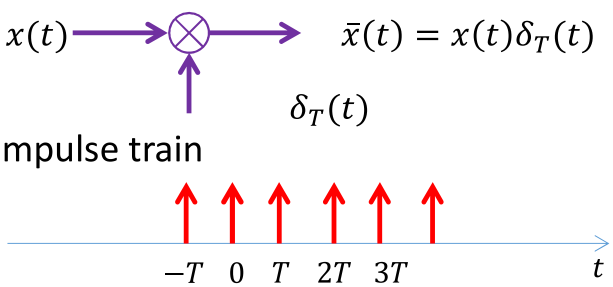


ω_c, denoted as the cut off freq of the lowpass filter has to be such that the original signal is fully recovered. $B_0 \le \omega_c$, $\omega_c \le 2\omega_0 - B_0$

The Amplitude Modulated (AM radio) carrier frequency ω_0 is in the frequency range 535-1605 kHz. Each carrier frequency is assigned at 10 kHz intervals.

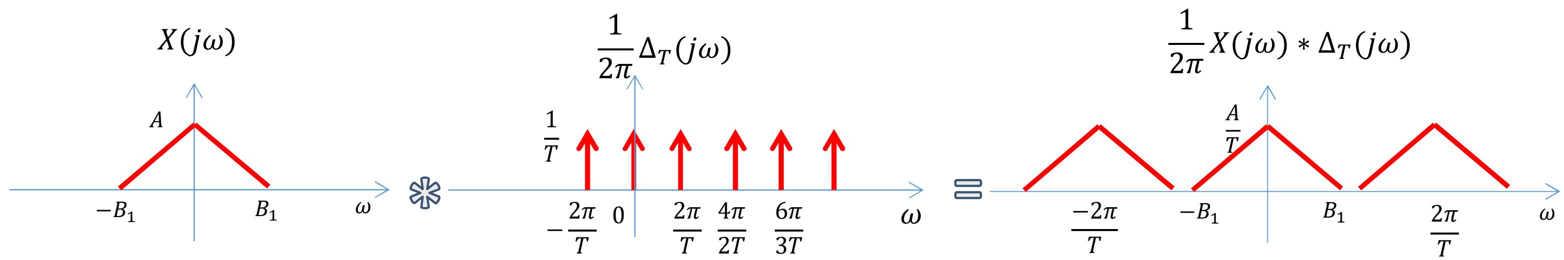
Sampling



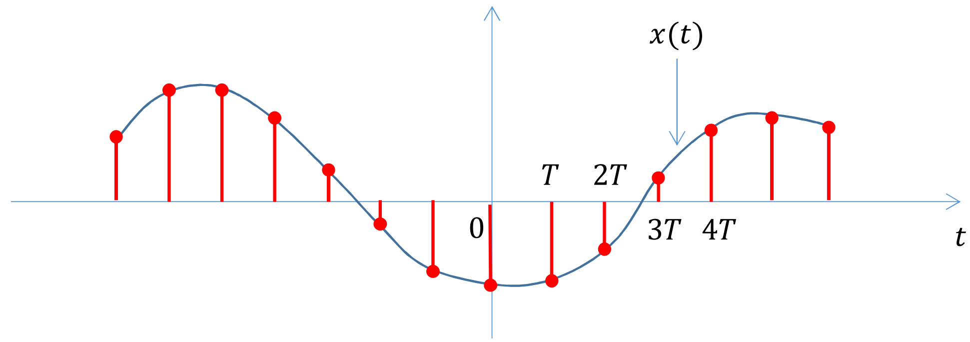


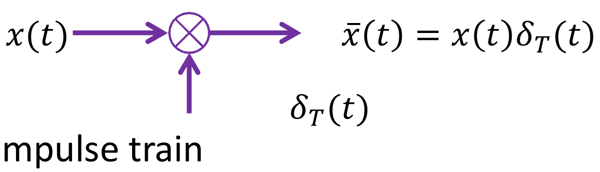
$$x(t)\delta_T(t) \xrightarrow{FT} \frac{1}{2\pi} X(j\omega) * \Delta_T(j\omega)$$

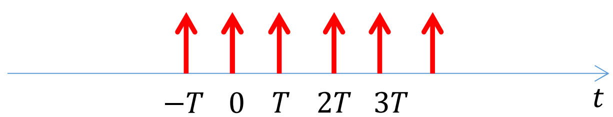
$$\begin{aligned}\overline{x}(t) &= x(t) \times \delta_T(t)\\ &= x(t) \sum \delta(t - nT)\\ &= \sum x(t)\delta(t - nT)\\ &= \sum x(nT)\delta(t - nT)\end{aligned}$$



Sampling







$$x(t)\delta_T(t) \xrightarrow{FT} \frac{1}{2\pi}X(j\omega) * \Delta_T(j\omega)$$

Sampling Theorem: Only if the sampling is fast enough ($\frac{2\pi}{T} \geq 2B_1$), the original signal $x(t)$ can be recovered from the sampled $\bar{x}(t)$ through a lowpass filter.